1. (a) Let $X$ be the outcome of tossing a fair die. What is the pgf $P_X(s)$ of $X$?
(b) Toss a die repeatedly. Let $a_n$ be the number of ways in which it is possible to arrive at exactly $n$ as the sum of the faces of the die. So, $a_1 = 1$ since $n = 1$ can be achieved in only 1 way, i.e., by getting a 1 in the first toss. Also, $a_3 = 4$ since there are four ways to arrive at 3 as the sum: 3, 2+1, 1+2, 1+1+1. Find the generating function of $a_n$. Hint: recursion.

2. Let $X_n$ ($n \geq 1$) be iid Bernoulli random variables with
\[ P[X_i = 1] = p = 1 - P[X_i = 0] \] (1)
and let $S_n = X_1 + \ldots + X_n$.
(a) Show that
\[ P[S_{n+1} = k] = pP[S_n = k - 1] + (1 - p)P[S_n = k]. \] (2)
(b) Multiply equation (2) by $s^k$ to obtain a recursion formula for the probability generating function of $S_n$ and solve it.
(c) Alternatively to 2b, derive a formula for the pgf of $S_n$ using $P_X(s)$.
(d) Using the pgf, verify that $S_n$ has a Binomial distribution, i.e., $S_n \sim b(k; n, p)$.

3. Let $X_n$ ($n \geq 1$) and $N$ be non-negative integer valued random variables, all with finite mean and finite variance. Assume that $X_n$ ($n \geq 1$) are iid, and independent of $N$.
Let $S_n = X_1 + \ldots + X_n$. Using pgf, verify that
\[ \text{Var}(S_N) = \text{E}[N]\text{Var}(X_1) + (\text{E}[X_1])^2\text{Var}(N) \] (3)

4. Let $X$ and $Y$ be jointly distributed non-negative integer valued random variables. For $|s| < 1$ and $|t| < 1$ define
\[ P_{X,Y}(s, t) := \sum_{j \geq 0, k \geq 0} s^j t^k P[X = j, Y = k]. \] (4)
Prove that $X$ and $Y$ are independent if and only if
\[ P_{X,Y}(s, t) = P_X(s)P_Y(t) \] (5)
for $|s| < 1$ and $|t| < 1$. Hint: Use Fubini to convert the right side into one double-indexed sum and compare coefficients.