8. A Markov chain has state space $S = \{1, 2, \ldots, 8\}$. Starting from $X_0 = 1$, the chain moves in each step from its current state $j$ to any of the larger states $\{k : k > j\}$ with equal probability. State 8 is absorbing.

(a) Compute the transition matrix.
(b) Decompose the state space into recurrent and transient classes.
(c) Find the expected number of steps to reach state 8.

9. Let $X_n$ be a Markov Chain.

(a) Assume that $f$ is a given 1-1 function of the state space, i.e., $f$ is invertible. Show that the sequence of random variables $f(X_n)$ form a Markov Chain as well.
(b) Show that this is not necessarily true if $f$ is not invertible. Hint: Consider $f(x) = x^2$ or $f(x) = |x|$ and an MC with only few states.

10. Let $P = [p_{ij}]_{i,j}$ denote the transition matrix of an MC. Define $q_{ij} = 1$ if $p_{ij} \neq 0$ and $q_{ij} = 0$ else. The matrix $Q = [q_{ij}]_{i,j}$ indicates whether it is possible to reach $j$ from $i$ in 1 step.

(a) Show that the matrix $Q^2$ indicates in how many ways it is possible to reach $j$ from $i$ in 2 steps.
(b) Assume that $S$ has $m$ states. Explain how the matrix $Q + Q^2 + \ldots + Q^m$ can be used to decide whether $j$ is reachable from $i$ or not.

11. (a) Assume that there exists an integer $n$ such that $p_{ij}^{(n)} \neq 0$ for all $i, j \in S$. Show that the MC is then irreducible.
(b) *Bonus question* The reverse is not true. Give a simple counter example.