12. Assume that the set $T$ of transient states of an MC is finite and set $m := |T|$.

(a) Argue that $T$ can not be closed. (Hint: use a basic result from class.)

(b) Conclude that $a := \max_{i \in T} \sum_{k \in T} P_{ik}^{(m+1)}$ is strictly less than one.

(c) Give a simple rough upper bound for the exit time from $T$ $\tau_{Tc} \geq n$ in terms of $a$.

13. The Media Police have identified six states associated with television watching: 0 (never watching), 1 (watch occasionally), 2 (watch frequently), 3 (addict), 4 (undergoing behavioral modification), 5 (brain dead). Transitions from state to state can be modelled as an MC with the following transition matrix:

$$
P = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
.5 & 0 & .5 & 0 & 0 & 0 \\
.1 & 0 & .5 & .3 & 0 & .1 \\
0 & 0 & 0 & .7 & .1 & .2 \\
1/3 & 0 & 0 & 1/3 & 1/3 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
$$

(a) Which states are transient, which are recurrent?

(b) Give the canonical decomposition of the state space.

(c) Set $q_i = P[X_n = 5$ for some $n \geq 1 | X_0 = i]$ Starting from state 1, we are interested in the chance to enter state 5 before state 0. Show that this chance is exactly equal to $q_1$.

(d) Using this fact, express this chance in terms of the limiting distribution.

(e) Looking at the rows of $P$ find four relations between $q_1, \ldots, q_4$, e.g., from row 3 we get $q_2 = .5q_2 + .3q_3 + .1$. Solve for $q_1$.

14. Let $S_n$ denote a simple random walk: $S_n = X_1 + \ldots + X_n$ with $X_n$ i.i.d. and $P[X_n = 1] = 1 - P[X_n = -1] = p = 1 - q$.

(a) Show that this chain is irreducible, i.e., find for every pair of states $i, j$ an integer $n$ such that $p_{ij}^{(n)} \neq 0$ (the exact value is not needed). Hint: distinguish $i > j$ and $i \leq j$.

(b) Based on known results on the return to zero from earlier homework decide whether 0 is recurrent or transient. Hint: your answer will dependent on the parameter $p$.

(c) *Bonus question* Recall Stirling's formula which implies that

$$
\binom{2n}{n} \approx \frac{4^n}{\sqrt{\pi n}}
$$

as well as the well known relation between geometric and arithmetic means which implies that $pq \leq 1/4$ with equality if and only if $p = q = 1/2$. Now, approximate the probability of passing from zero to zero in $2n$ steps $P_{00}^{(2n)}$ using Stirling’s formula and determine, for which values of the parameter $p$ the sum

$$
\sum_{k=1}^{\infty} P_{00}^{(k)}
$$

converges. Conclude whether 0 is recurrent or transient depending on $p$, thus obtaining the same result as before.