STAT 552 Homework 4

Due date: In class on Thursday, September 29, 2005

Instructor: Dr. Rudolf Riedi

12. Assume that the set T of transient states of an MC is finite and set $m := |T|$.
    (a) Argue that T can not be closed. (Hint: use a basic result from class.)
    (b) Conclude that $a := \max_{i \in T} \sum_{k \in T} p_{ik}^{(m+1)}$ is strictly less than one.
    (c) Give a simple rough upper bound for the exit time from T $P[\tau_{T^c} \geq n]$ in terms of a.

13. The Media Police have identified six states associated with television watching: 0 (never watching), 1 (watch occasionally), 2 (watch frequently), 3 (addict), 4 (undergoing behavioral modification), 5 (brain dead). Transitions from state to state can be modelled as an MC with the following transition matrix:

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 & 0 \\ .1 & 0 & .5 & .3 & 0 & .1 \\ 0 & 0 & 0 & .7 & .1 & .2 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{1}$$

    (a) Which states are transient, which are recurrent?
    (b) Give the canonical decomposition of the state space.
    (c) Set $q_i = P[X_n = 5 \text{ for some } n \geq 1 \mid X_0 = i]$. Starting from state 1, we are interested in the chance to enter state 5 before state 0. Show that this chance is exactly equal to q_1.
    (d) Using this fact, express this chance in terms of the limiting distribution.
    (e) Looking at the rows of P find four relations between $q_1, \ldots, q_4$, e.g., from row 3 we get $q_2 = .5q_2 + .3q_3 + .1$. Solve for q_1.

14. Let S_n denote a simple random walk: $S_n = X_1 + \ldots + X_n$ with X_n i.i.d. and $P[X_n = 1] = 1 - P[X_n = -1] = p = 1 - q$.
    (a) Show that this chain is irreducible, i.e., find for every pair of states i, j an integer n such that $p_{ij}^{(n)} \neq 0$ (the exact value is not needed). Hint: distinguish $i > j$ and $i \leq j$.
    (b) Based on known results on the return to zero from earlier homework decide whether 0 is recurrent or transient. Hint: your answer will dependent on the parameter p.
    (c) *Bonus question* Recall Stirling's formula which implies that

$$\begin{pmatrix} 2n \\ n \end{pmatrix} \simeq \frac{4^n}{\sqrt{\pi n}} \tag{2}$$

    as well as the well known relation between geometric and arithmetic means which implies that $pq \leq 1/4$ with equality if and only if $p = q = 1/2$. Now, approximate the probability of passing from zero to zero in $2n$ steps $p_{00}^{(2n)}$ using Stirling's formula and determine, for which values of the parameter p the sum

$$\sum_{k=1}^{\infty} p_{00}^{(k)} \tag{3}$$

    converges. Conclude whether 0 is recurrent or transient depending on p, thus obtaining the same result as before.