STAT 552 PRACTICE TEST

This test is an **individual** exam.
It is **closed books**. All personal notes are allowed.

Time: 180 Minutes.

Given as a Homework 5, due on Thursday, October 20rd, 2005

Instructor: Dr. Rudolf Riedi

Total 60 points. Spend roughly at the most a quarter hour per 5 points.

1. (25 points)

   Let $0 \leq s < S$ be two integer parameters. Let D_n be a sequence of i.i.d. random variables with $p_k = P[D_n = k] > 0$ for all $k \geq 0$ and $P[D_n = \infty] = 0$. Suppose $X_0 \leq S$. Recall that $(u)_+ = \max(u, 0) = (u + |u|)/2$ and define

   $$X_n := \begin{cases} (X_{n-1} - D_n)_+ & \text{if } s < X_{n-1} \leq S, \\ (S - D_n)_+ & \text{if } X_{n-1} \leq s. \end{cases} \tag{1}$$

   You may think of X_n as tracking the stock (number of items) in a store at the end of the nth day, where D_n is the demand on the n-th day. If the stock falls below s in the evening it is replenished to S over night.

   (a) (2 points) Argue in a short sentence that X_n forms a Markov Chain.

   (b) (3 points) Determine its equivalence classes.

   (c) (5 points) Compute the long run average stock level $(X_0 + \ldots + X_{N-1})/N$ in terms of the stationary distribution.

   (d) (3 points) For the remainder let us consider a specific example. Let $s = 0$ and $S = 2$. Let $p_0 = 1/2$, $p_1 = 2/5$ and $p_2 = 1/10$. Compute the transition matrix $\boldsymbol{P}$.

   (e) (3 points) For this example show that the stationary distribution is $(5/18, 8/18/5/18)$.

   (f) (4 points) For the same example compute the long run fraction of periods of unsatisfied demand, i.e., the long run fraction of days with $X_n = 0$.

2. (15 points)

   Let Z_n be a simple branching process with $Z_0 = 1$, $Z_1 = Z_{1,1}$ etc. and i.i.d. $Z_{j,k}$. As usual, let π denote the probability of extinction.

   (a) (7 points) Show that if $P[Z_{j,k} = 0] = 0$ then $\pi = 0$. (Hint: A moments thought reveals a very simple argument without computation).

   (b) (8 points) Show the reverse, i.e., if $\pi = 0$ then necessarily $P[Z_{j,k} = 0] = 0$.

3. (15 points)

   Consider a simple random walk S_n with $p > 1/2$. In class, we have seen that this MC is irreducible and transient. Here, you are asked to derive this result in a different manner than in class.

   (a) (7 points) Use the strong law of large numbers to conclude that $P[S_n \to \infty] = 1$.

   (b) (8 points) Use this fact to show that $P[\tau_0^{(1)} = \infty] > 0$. (Hint: indirect argument; use the dissection principle.)

   You may answer question (b) independently from (a), assuming the (a) is true.

4. (5 points) Let j be an absorbing state of a Markov Chain. Which of the following is true?

   (a) State j necessarily transient,

   (b) state j necessarily recurrent,

   (c) state j could be either.

   If your answer is one of the first two options, provide an argument; if your answer is the third option, provide two Chains, one with an absorbing transient state and one with an absorbing recurrent state.