STAT 552 Homework 9

Due: Tuesday, November 29, 2005

Instructor: Dr. Rudolf Riedi

27. (On-off system)

Consider a machine of a given type which stay "operative" for an exponential length of time of mean duration $1/\lambda$. When a breakdown occurs, repairs are started immediately and last for an exponential length of time of mean duration $1/\mu$. Repairs return the machine into the original "operative" state. Let Z_t be the state of the machine at time t.

(a) Explain why we may write Z_t as a continuous time Markov chain with state space $S = \{0, 1\}$, holding time parameters $\lambda(0) = \lambda$ and $\lambda(1) = \mu$ and transition matrix of the underlaying discrete Markov chain as

$$Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

(b) Give the forward and backward equations $P'(t) = P(t)A = AP(t)$, in other words, compute $A = P'(0)$ from a result in class.

(c) Using the fact that $P(0)$ is the identity matrix, show that

$$P'_{00}(t) = \mu - (\mu + \lambda)P_{00}(t).$$

Hint: eliminate the other $P'_{ij}(t)$ in the backward or forward equation.

(d) Show that this differential equation is solved by

$$P_{00}(t) = c \exp(-(\mu + \lambda)t) + \frac{\mu}{\lambda + \mu}.$$

(e) Compute the matrix $P(t)$. Hint: use the initial condition $P(0)$.

Note, that if the machine starts in "operative" state, $P_{00}(t)$ gives the probability that it is operative at time t (it might have been repaired several times in between).

(f) At time $t = 0$, N machines of this type are placed independently in use and all are operative. Show that the number of machines operative at time $t > 0$ has a binomial distribution with success parameter

$$p = \frac{1}{\lambda + \mu} (\mu + \lambda \exp(-(\mu + \lambda)t))$$