1. (a) Let $X$ be the outcome of tossing a fair die. What is the pgf $P_X(s)$ of $X$?

(b) Toss a die repeatedly. Let $a_n$ be the number of ways in which it is possible to arrive at exactly $n$ as the sum of the faces of the die. So, $a_1 = 1$ since $n = 1$ can be achieved in only 1 way, i.e., by getting a 1 in the first toss. Also, $a_3 = 4$ since there are four ways to arrive at 3 as the sum: 3, 2 + 1, 1 + 2, 1 + 1 + 1. Find the generating function of $a_n$.

Hint: recursion.

2. Let $X_n (n \geq 1)$ be iid Bernoulli random variables with

$$P[X_i = 1] = p = 1 - P[X_i = 0]$$

(1)

and let $S_n = X_1 + \ldots X_n$.

(a) Show that

$$P[S_{n+1} = k] = pP[S_n = k - 1] + (1 - p)P[S_n = k].$$

(2)

(b) Multiply equation (2) by $s^k$ to obtain a recursion formula for the probability generating function of $S_n$ and solve it.

(c) Alternatively to 2b, derive a formula for the pgf of $S_n$ using $P_X(s)$.

(d) Using the pgf, verify that $S_n$ has a Binomial distribution, i.e., $S_n \sim b(k; n, p)$.

3. Let $X_n (n \geq 1)$ and $N$ be non-negative integer valued random variables, all with finite mean and finite variance. Assume that $X_n (n \geq 1)$ are iid, and independent of $N$.

Let $S_n = X_1 + \ldots X_n$. Using pgf, verify that

$$\text{Var}(S_N) = \mathbb{E}[N]\text{Var}(X_1) + (\mathbb{E}[X_1])^2\text{Var}(N)$$

(3)

4. Let $X$ and $Y$ be jointly distributed non-negative integer valued random variables. For $|s| < 1$ and $|t| < 1$ define

$$P_{X,Y}(s, t) := \sum_{j \geq 0, k \geq 0} s^j t^k P[X = j, Y = k].$$

(4)

Assume that $X$ and $Y$ are independent. Show that

$$P_{X,Y}(s, t) = P_X(s)P_Y(t)$$

(5)

for $|s| < 1$ and $|t| < 1$. Hint: Use Fubini to convert the right side into one double-indexed sum and compare coefficients. Alternatively, you can write $P_{X,Y}(s, t)$ as an expected value (of which random variable?).