5. In a branching process we have

\[ P(s) = P_{Z_{j,k}}(s) = as^2 + bs + c \]  
\[ \text{with } a > 0, b > 0, c > 0 \text{ and } P(1) = 1. \]

(a) Compute the extinction probability \( \pi \).

(b) Give a condition for sure extinction.

(c) What is the underlying distribution of offsprings \( Z_{j,k} \)?

6. Conduct a compound experiment as follows. Suppose, \( N \) is a non-negative integer valued random variable with \( q_k := P[N = k] \) and \( \sum_{k \geq 0} q_k = 1 \), i.e., \( P[N = \infty] = 0 \). Observe \( N \) items and mark each of the \( N \) items independently of each other and independently of the value of \( N \) with a probability \( p \), where \( 0 < p < 1 \). To be precise, set \( X_i = 1 \) with probability \( p \) indicating that the \( i \)th item is marked, and \( X_i = 0 \) otherwise. Let \( V \) denote the number of marked items among \( X_1, \ldots, X_N \).

(a) Relate \( V \) to the \( X_i \).

(b) Give a simple formula for \( P_V(s) \).

(c) What is the probability that all items are marked in terms of \( p, q_k \) and \( P_N(s) = \sum_{k \geq 0} q_k s^k \)?

Hint: The number of items is random! To avoid problems with "double-randomness" partition the event "all items marked" into sub-events where one the random variables is known. Compare to the derivation of the compound formula in class.

7. Suppose \( X \) is a non-negative integer valued random variable with \( P[X = \infty] = 0 \). Suppose \( T \) is a geometric random variable independent of \( X \), i.e., \( T \sim g(k; \theta) \) where the parameter \( \theta \) lies in \( (0, 1) \). To be precise, \( P[T \geq k] = \theta^k \) for \( k \geq 0 \).

(a) Verify that \( \sum_k P[T \geq k, X = k] = P_X(\theta) \).

(b) Compute \( P[T \geq X] \).

Hint: Again, there is "double-randomness". Split the event into sub-events where one the random variables is known.

8. Consider a simple branching process with \( P(s) = q + ps \). Let \( T \) denote the time of extinction, i.e., \( T = \inf \{ n \geq 1 : Z_n = 0 \} \).

(a) Express the event \( \{ T = n \} \) in terms of the events \( \{ Z_k = 0 \} \).

(b) Express \( P[T = n] \) in terms of \( P(s) \).

(c) Derive an explicit formula for \( P[T = n] \) in terms of \( p \) and \( q \). Hint: recursion.