STAT 552 Homework 3

Due date: In class on Tuesday, October 3, 2006

Instructor: Dr. Rudolf Riedi

9. A Markov chain has state space $S = \{1, 2, \ldots, 8\}$. Starting from $X_0 = 1$, the chain moves in each step from its current state j to any of the larger states $\{k : k > j\}$ with equal probability. State 8 is absorbing.

   (a) Compute the transition matrix.

   (b) Decompose the state space into recurrent and transient classes.

   (c) Find the expected number of steps to reach state 8.
   Hint: Set $N = \inf\{n \geq 0 : X_n = 8\}$ and $q_i = \mathbb{E}[N|X_0 = i]$ and show first the following:

   $$q_i = \mathbb{E}[N|X_0 = i] = \sum_{k=1}^{8} \mathbb{E}[N|X_0 = i, X_1 = k]p_{ik} = \sum_{k=1}^{8}(1 + q_k)p_{ik}. \tag{1}$$

   Here you need to justify both, steps 2 and 3. Finish by solving the recursion starting at q_8 which is easy (trivial) to compute. You should find $q_1 = 363/140 = 2.6$ (pocket calculator may be handy but not required).

10. Let X_n be a Markov Chain.

    (a) Assume that f is a given 1-1 function of the state space, i.e., f is invertible. Show that the sequence of random variables $f(X_n)$ form a Markov Chain as well.

    (b) Show that if f is not invertible then this can be true or false, depending on the chain and the function. Hint: Consider $f(x) = x^2$ or $f(x) = |x|$ and a small state space. Give an example of a MC X_n where $f(X_n)$ is not Markov, and one where $f(X_n)$ is Markov.

11. Let $\boldsymbol{P} = [p_{ij}]_{(ij)}$ denote the transition matrix of an MC. Define $q_{ij} = 1$ if $p_{ij} \neq 0$ and $q_{ij} = 0$ else. The matrix $\boldsymbol{Q} = [q_{ij}]_{(ij)}$ indicates whether it is possible to reach j from i in 1 step.

    (a) Show that the matrix $\boldsymbol{Q}^2$ indicates in how many ways it is possible to reach j from i in 2 steps.

    (b) Assume that S has m states. Explain how the matrix $\boldsymbol{Q} + \boldsymbol{Q}^2 + \ldots + \boldsymbol{Q}^m$ can be used to decide whether j is reachable from i or not.
    Hint: Using a recursive argument show that $\boldsymbol{Q}^k$ indicates in how many ways it is possible to reach j from i in k steps.

12. (a) Assume that there exists an integer n such that $p_{ij}^{(n)} \neq 0$ for all $i, j \in S$. Show that the MC is then irreducible (meaning that S forms class).

    (b) *Bonus question* The reverse is not true. Give a simple counter example.