STAT 552 Homework 4

Due date: In class on Tuesday, October 10, 2006

Instructor: Dr. Rudolf Riedi

General hint: the following fact is intuitively clear and can be used for the solutions without proof.
If $i \to j$ then there exists a sequence $i_0, \dots, i_m$ such that
$i_0 = i$, $i_m = j$, all i_k are different and $p_{i_k i_{k+1}} \neq 0$ for all $k = 0, \dots m-1$.

13. Set T to be the set of transient states of a MC X_n and set $m := |T|$ ($m = \infty$ if T is not finite).

    (a) Let i be an arbitrary state in S, and let $A := \{j \in S : i \to j\}$ be the set of all states that can be reached from i. Show that A is closed.

    (b) Assume that T is finite. Argue that T can not be closed. Hint: Don't look for the answer in (13a).

    (c) [Way to leave finite T] Assume that T is finite. Show that for any $i \in T$ there exists $k \notin T$ and n such that $p_{ik}^{(n)} \neq 0$.
    Hints: It is easy to show that there must be recurrent states in S—the task is to show that there is one that can be reached from i. Use (13a).

    (d) [Way to leave finite T in bounded time] Improve (13c) as follows. Assume that T is finite. Show that for any $i \in T$ there exists $k \notin T$ such that $p_{ik}^{(m+1)} \neq 0$. Hint: "General hint".

    Note: from all this we may conclude that the sojourn time in finite T decays faster than exponential:

    $$P[\tau_{T^c} \geq u(m+1)+1] \leq m a^u.$$

14. The Media Police have identified six states associated with television watching: 0 (never watching), 1 (watch occasionally), 2 (watch frequently), 3 (addict), 4 (undergoing behavioral modification), 5 (brain dead). Transitions from state to state can be modelled as an MC with the following transition matrix:

    $$P = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ .5 & 0 & .5 & 0 & 0 & 0 \\ .1 & 0 & .5 & .3 & 0 & .1 \\ 0 & 0 & 0 & .7 & .1 & .2 \\ 1/3 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad (1)$$

    (a) Which states are transient, which are recurrent?

    (b) Give the canonical decomposition of the state space.

    (c) Set $q_i = P[X_n = 5 \text{ for some } n \geq 1 \,|X_0 = i]$. Starting from state 1, we are interested in the chance to enter state 5 before state 0. Show that this chance is exactly equal to q_1.

    (d) Looking at the rows of P find four relations between $q_1, \dots, q_4$, e.g., from row 3 we get $q_2 = .5q_2 + .3q_3 + .1$. Solve for q_1. Hint: recall problem 9.

    (e) (Bonus) Express q_1 in terms of the limiting distribution.

15. Let Z_n be a simple branching process with $Z_0 = 1$, $Z_1 = Z_{1,1}$ etc. and i.i.d. $Z_{j,k}$. As usual, let π denote the probability of extinction.

    (a) Show that if $P[Z_{j,k} = 0] = 0$ then $\pi = 0$.

    (b) Show the reverse, i.e., if $\pi = 0$ then necessarily $P[Z_{j,k} = 0] = 0$. Hint: both are simple.