STAT 552 Homework 5

Due date: In class on Thursday, October 19, 2006

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16. Let S_n denote a simple random walk: $S_n = X_1 + \ldots + X_n$ with X_n i.i.d. and $P[X_n = 1] = 1 - P[X_n = -1] = p = 1 - q$.

    (a) Show that this chain is irreducible, i.e., find for every pair of states i, j an integer n such that $p_{ij}^{(n)} \neq 0$ (the exact value is not needed). Hint: distinguish $i > j$ and $i \leq j$.

    (b) Based on known results on the return to zero from earlier homework decide whether 0 is recurrent or transient. Hint: your answer will dependent on the parameter p.

    (c) Assume $p > 1/2$. Use the strong law of large numbers to conclude that $P[S_n \to \infty] = 1$.

    (d) Assume that 0 is recurrent. Show that $P[S_n \to \infty] = 1$ is impossible.

17. Let $0 \leq s < S$ be two integer parameters. Let D_n be a sequence of i.i.d. random variables with $p_k = P[D_n = k] > 0$ for all $k \geq 0$ and $P[D_n = \infty] = 0$. Suppose $X_0 \leq S$. Recall that $(u)_+ = \max(u, 0) = (u + |u|)/2$ and define

$$X_n := \begin{cases} (X_{n-1} - D_n)_+ & \text{if } s < X_{n-1} \leq S, \\ (S - D_n)_+ & \text{if } X_{n-1} \leq s. \end{cases} \tag{1}$$

    You may think of X_n as tracking the stock (number of items) in a store at the end of the nth day, where D_n is the demand on the n-th day. If the stock falls below s in the evening it is replenished to S over night.

    (a) Argue in a short sentence that X_n forms a Markov Chain.

    (b) Determine its equivalence classes.

    (c) Compute the long run average stock level $(X_0 + \ldots + X_{N-1})/N$ in terms of the stationary distribution.

    (d) For the remainder let us consider a specific example. Let $s = 0$ and $S = 2$. Let $p_0 = 1/2$, $p_1 = 2/5$ and $p_2 = 1/10$. Compute the transition matrix $\boldsymbol{P}$.

    (e) For this example show that the stationary distribution is $(5/18, 8/18/5/18)$.

    (f) For the same example compute the long run fraction of periods of unsatisfied demand, i.e., the long run fraction of days with $X_n = 0$.

18. Let j be an absorbing state of a Markov Chain. Which of the following is true?

    (a) State j necessarily transient,

    (b) state j necessarily recurrent,

    (c) state j could be either.

    If your answer is one of the first two options, provide an argument; if your answer is the third option, provide two Chains, one with an absorbing transient state and one with an absorbing recurrent state.