STAT 552 HOMEWORK 6

To reduce load his homework is voluntary and will not be graded. It is meant as an opportunity to practice.

PRACTICE TEST 1

This test is an **individual** exam.
It is **closed books**. All personal notes as well as the lecture notes and solution sheets are allowed.

Time: 180 Minutes. Total 60 points, (15 minutes per 5 points)

Instructor: Dr. Rudolf Riedi

When done with the test, fill out this page.

Your name:..

State the time during which you took the test:..............................

Sign with the Honor Code:

Points (to be filled by grader):

	1	2	3	4	Total
Score					
Out of	10	20	20	10	60

1. (10 points) Let X_n be a Markov chain (MC) with two states, say 0 and 1. The state 1 is absorbing. You are asked to study the nature of the other state 0 (recurrent or transient) as follows.

   (a) (5 points) Give an example of such an MC where the other state 0 is recurrent as well or show that this is impossible, i.e., show that 0 must be transient.

   (b) (5 points) Give an example of such an MC where the other state 0 is transient or show that this is impossible, i.e., show that 0 must be recurrent as well.

   In order to provide examples of Markov chains simply give the transition matrix.

2. (20 points) Let B_n $(n \geq 0)$ denote i.i.d. Bernoulli variables where success $(B_n = 1)$ and failure $(B_n = 0)$ occur with equal probability 1/2. For $n \geq 1$ let

$$X_n = 3B_{n-1} + 2B_n \tag{1}$$

   (a) (10 points) Show that X_n is a Markov Chain with state space $\{0, 2, 3, 5\}$ and compute its transition matrix $\boldsymbol{P}$.

   (b) (5 points) Compute $\boldsymbol{P}^2$. Hint: you may use matrix multiplication $\boldsymbol{P} \cdot \boldsymbol{P}$ or compute the 2-stage transition probabilities $P[X_2 = k | X_0 = i]$.

   (c) (5 points) Compute the transient and recurrent classes.

   (d) (Bonus: 5 points) Is $Y_n := B_{n-1} + B_n$ a Markov chain as well? [show your argument]

3. (20 points) A boy and a girl move to the same two-bar-town on the same day. Each night, boy visits one bar, starting the first night with bar 1 and continues by selecting a bar for the next night according to a Markov chain with transition matrix $\boldsymbol{P}$ below. Similarly, girl visits one bar a night, starting with bar 2 and selecting the next bar according to $\boldsymbol{Q}$ below.

$$\boldsymbol{P} = \begin{pmatrix} .8 & .2 \\ .2 & .8 \end{pmatrix} \qquad \boldsymbol{Q} = \begin{pmatrix} .1 & .9 \\ .9 & .1 \end{pmatrix}$$

   Once girl and boy go to same bar one night, they keep going to the same bar (they go together to either bar 1 or bar 2) for all future for the obvious reason.

   (a) (5 points) Model the progress of boy and girl finding each other as a single Markov chain where only one (!) state is absorbing. Compute the transition matrix for this chain.

   (b) (5 points) Let N denote the number of the night when girl meets boy. Compute the expected number of nights $\mathbb{E}[N]$ it takes for boy and girl to meet. Hint: Either compute the distribution of N, or develop an invariance equation for $\mathbb{E}[N]$.

   (c) (10 points. can be answered without solving the previous questions) Assume that once boy and girl have met, they visit the bars according to the transition matrix $\boldsymbol{Q}$. What is the long-term-average of times they visit bar 1? (For clarification we note that the long-term-averages of bar 1 and bar 2 add up to 1). What is the long-term average of visits if they follow the matrix $\boldsymbol{P}$?

4. (10 points)

   Consider the renewal process N_t where the interarrival times Y_0, Y_i $(i = 1, 2, ...)$ take only the values 1 or 2 and are all i.i.d. with common distribution:

$$P[Y_k = 2] = 1 - P[Y_k = 1] = p \quad (k = 0, 1, 2, ...)$$

   Clearly, N_t will jump only at integer locations. Consider a fixed integer $t \geq 1$.

   (a) (5 points) Find the k for which the event $\{N_t = k\}$ is empty.

   (b) (5 points) For all other k show that

$$P[N_t = k] = \binom{k}{t-k} p^{t-k}(1-p)^{2k-t} + \binom{k}{t-k-1} p^{t-k}(1-p)^{2k-t+1}$$

   Hint: distinguish whether $S_{k-1} = t$ or $t - 1$.