23. Consider the renewal process $N(t)$ where the interarrival times $Y_0, Y_i \ (i = 1, 2, \ldots)$ take only the values 1 or 2 and are all i.i.d. with common distribution:

$$P[Y_k = 2] = 1 - P[Y_k = 1] = p \quad (k = 0, 1, 2, \ldots)$$

Recall the formula obtained in an earlier homework:

$$P[N(t) = k] = \left(\frac{k}{t-k}\right) p^{t-k}(1-p)^{2k-t} + \left(\frac{k}{t-k-1}\right) p^{t-k}(1-p)^{2k-t+1}$$

(a) Write the renewal equation for $\mu(t) = V(t) = E[N(t)]$.

(b) Show that $\mu(0) = 0$ and $\mu(1) = 1 - p$ with the $Y_0$ given.

(c) Verify that $\mu(n) = n/(1+p) - p^2(1 - (-p)^n)/(1+p)^2$ for integer $n \geq 0$.

(d) Compute $\lim_{n \to \infty} \mu(n)/n$.

(e) Show that the sequence $X_n = N(n)$ is not a Markov chain.

(f) Show that the sequence $S_n = Y_0 + \ldots + Y_n$ is a Markov chain.

(g) Compute $F_0$, i.e., the distribution for $Y_0$ which makes the renewal sequence stationary.

(h) Assume now that $Y_0$ is distributed according to $F_0$. Compute $\mu(0)$ and $\mu(1)$ in this case.

(i) Is $X_n = N(n)$ a Markov chain when $Y_0$ is distributed according to $F_0$?

24. (Age and excess life)

Let $N(t)$ be a pure renewal process. Let $A(t)$ be the age of the current item, and $B(t)$ its excess life time as usual. Also, set as usual

$$F_0(x) = \frac{1}{\mu} \int_0^x (1 - F(u)) \, du.$$  \hfill (1)

(a) Fix $x$. State the renewal equation for $Z(t) = P[B(t) > x]$.

(b) Recall Blackwell’s key renewal theorem which says that $U * z(t) \to \frac{1}{\mu} \int_0^\infty z(u) \, du$. Compute $\lim Z(t)$ with $Z$ as in (a).

(c) From (a) and (b) conclude that $P[B(t) \leq x] \to F_0(x)$ as $t \to \infty$.

(d) In the same way, show that $P[A(t) \leq x] \to F_0(x)$.

(e) Fix now two positive numbers $x$ and $y$ and define

$$\tilde{Z}(t) := P[A(t) > x, B(t) > y].$$  \hfill (2)

Show that $\tilde{Z}(t)$ satisfies a renewal equation with $\tilde{z}(t) = 1_{ \{t>x\} } (1 - F(t+y))$.

Hint: Write

$$\tilde{Z}(t) = P[A(t) > x, B(t) > y, Y_1 > t] + P[A(t) > x, B(t) > y, Y_1 \leq t].$$

and show that the first term is $\tilde{z}$, while the second terms is $\tilde{Z} \ast F$.

(f) Compute $\lim_{t \to \infty} \tilde{Z}(t)$.

Hint: Use Blackwell. You should find $1 - F_0(x+y)$.