STAT 552 Homework 7

Due date: In class on Thursday, November 16, 2006

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23. Consider the renewal process $N(t)$ where the interarrival times Y_0, Y_i $(i = 1, 2, ...)$ take only the values 1 or 2 and are all i.i.d. with common distribution:

$$P[Y_k = 2] = 1 - P[Y_k = 1] = p \quad (k = 0, 1, 2, ...)$$

Recall the formula obtained in an earlier homework:

$$P[N(t) = k] = \binom{k}{t-k} p^{t-k}(1-p)^{2k-t} + \binom{k}{t-k-1} p^{t-k}(1-p)^{2k-t+1}$$

   (a) Write the renewal equation for $\mu(t) = V(t) = \mathbb{E}[N(t)]$.
   (b) Show that $\mu(0) = 0$ and $\mu(1) = 1 - p$ with the Y_0 given.
   (c) Verify that $\mu(n) = n/(1+p) - p^2(1 - (-p)^n)/(1+p)^2$ for integer $n \geq 0$.
   (d) Compute $\lim_{n\to\infty} \mu(n)/n$.
   (e) Show that the sequence $X_n = N(n)$ is not a Markov chain.
   (f) Show that the sequence $S_n = Y_0 + ... + Y_n$ is a Markov chain.
   (g) Compute F_0, i.e., the distribution for Y_0 which makes the renewal sequence stationary.
   (h) Assume now that Y_0 is distributed according to F_0. Compute $\mu(0)$ and $\mu(1)$ in this case.
   (i) Is $X_n = N(n)$ a Markov chain when Y_0 is distributed according to F_0?

24. (Age and excess life)

Let $N(t)$ be a *pure* renewal process. Let $A(t)$ be the age of the current item, and $B(t)$ its excess life time as usual. Also, set as usual

$$F_0(x) = \frac{1}{\mu} \int_0^x (1 - F(u))du. \tag{1}$$

   (a) Fix x. State the renewal equation for $Z(t) = P[B(t) > x)]$.
   (b) Recall Blackwell's key renewal theorem which says that $U * z(t) \to \frac{1}{\mu} \int_0^\infty z(u)du$. Compute $\lim Z(t)$ with Z as in (a).
   (c) From (a) and (b) conclude that $P[B(t) \leq x] \to F_0(x)$ as $t \to \infty$.
   (d) In the same way, show that $P[A(t) \leq x] \to F_0(x)$.
   (e) Fix now two positive numbers x and y and define

$$\tilde{Z}(t) := P[A(t) > x, B(t) > y]. \tag{2}$$

   Show that $\tilde{Z}(t)$ satisfies a renewal equation with $\tilde{z}(t) = \mathbf{1}_{[t>x]}(1 - F(t + y))$.
   Hint: Write

$$\tilde{Z}(t) \quad = \quad P[A(t) > x, B(t) > y, Y_1 > t] + P[A(t) > x, B(t) > y, Y_1 \leq t].$$

   and show that the first term is $\tilde{z}$, while the second terms is $\tilde{Z} * F$.
   (f) Compute $\lim_{t\to\infty} \tilde{Z}(t)$.
   Hint: Use Blackwell. You should find $1 - F_0(x + y)$.