STAT 552 Homework 8

Due date: In class on Tuesday, November 21, 2006

Instructor: Dr. Rudolf Riedi

24. (Age and excess life, continued)

    Set $\bar{F}_0(u) = 1 - F_0(u)$ for simplicity.

    (g) Assume $A(t)$ and $B(t)$ are independent for all $t > 0$. Show that $\bar{F}_0(x+y) = \bar{F}_0(x)\bar{F}_0(y)$. (Use results from last homework)

    (h) Assume that $\bar{F}_0(x+y) = \bar{F}_0(x)\bar{F}_0(y)$. Show that $F_0(x) = 1 - \exp(-\alpha x)$ for some α.
    Hint: Write $g(x) = -\log(\bar{F}_0(x))$. You can cite an argument from class where we showed that under certain assumptions $V(t)$ must be linear ($V(t) = t/\mu$). There is no need to repeat that argument here. You need to make sure, though, that g satisfies the same assumptions as the function V in that argument.

    (i) Derive a formula for F in terms of F_0.
    Hint: Take a derivative in the equation which defines F_0 in terms of F.

    (j) Assume that $F_0(x) = 1 - \exp(-\alpha x)$. Show that $F = F_0$.

    (k) Conclude the following: If $A(t)$ and $B(t)$ are independent for all $t > 0$ then the renewal process necessarily has exponential interarrival times, i.e., it is the Poisson Renewal Process.

25. (Sample paradox)

    Consider a Poisson renewal process $N(t)$ with exponential interarrival times $Y_i \sim \exp(\alpha)$ ($i = 0, 1, ...$). Fix $t > 0$. Let $C := C(t) := A(t) + B(t)$ be the time between the last arrival before t and the next arrival after t.

    Show that C is statistically larger than Y_i, meaning that $P[C > x] \geq P[Y_i > x]$. In other words, C assumes large values more often than Y_i. Show also, that $\mathbb{E}[C] > \mathbb{E}[Y_i] = 1/\alpha$.
    Hint: Recall the distribution of $B(t)$

    Remark This fact is called the "sampling paradox". It has the following practical implication. Assume that we attempt to estimate the mean arrival rate α as $1/\mathbb{E}[C]$. To this end, we estimate $\mathbb{E}[C]$ as the sample mean $(C(t_1) + ... + C(t_n))/n$ for some fixed, pre-determined times t_i (for instance set t_i to be the i-th day in a month). Then we have a bias towards underestimating α. The "sampling paradox" says, in a way, that we are more likely to see large waiting times. Indeed, it is more difficult for a short waiting time to occur exactly during our predetermined fixed sample instants t_i.

    As a practical example picture light bulbs that are immediately replaced when they burn out. If we check on every 15th of every month how long the current bulb is holding out (measure $C(t_i)$) then we are more likely to see a long lasting bulb, and less likely to see by chance one of the poor bulbs that burn out within hours.