26. (Sums of Poisson Point Processes)

(a) Let $N$ and $M$ be two PRM in same space $E$ with intensity measures $\nu$ and $\mu$, respectively, and independent of each other. Show that $K(A) = N(A) + M(A)$ is again a PRM with intensity $\kappa = \nu + \mu$.

Hint: Assume that the points of $N$ are $\{X_n\}$ and the points of $M$ are $\{Y_n\}$ and determine first what the points of $K$ are.

(b) Assume now that $N$ and $M$ are PRMs on the positive real line, i.e., on $E = \mathbb{R}^+$ and that both are homogeneous, i.e., $\nu$ and $\mu$ are both Lebesgue measure: $\mu(a, b) = \nu(a, b) = b - a$ for $b > a > 0$. You may assume also, that the points of $N$, $M$ and $K$ are ordered, thus, the output of a renewal process with exponential interarrival times of mean 1, 1 and $1/2$, respectively. Explain why this shows that the minimum of iid mean 1 exponential random variables is again an exponential r.v., with mean $1/2$.

27. Customer arrival times in a bank form a homogeneous PRM with intensity $\mu(dt) = \alpha dt$ on the interval $[9, 16]$, a subset of $\mathbb{R}^+$. The each require a service time with exponential distribution $\sim \text{exp}(\lambda)$. We assume that service starts immediately at arrival (no waiting time...what a wonderful life). How many clients on average are still being serviced when the bank closes at 4pm, i.e., at time point $16$?

Your solution should be $\frac{\alpha}{\lambda} \left[1 - \exp(-\lambda(b-a))\right]$ with $b = 16$, $a = 9$

28. Let $w$ denote a (deterministic) exponential function, i.e., $w(t) = \exp(-\theta t)$ for $t \geq 0$ and zero else, where $\theta > 0$ denotes a given parameter. Given points $\{S_n\}$ set

$$X(t) = \sum_{n \geq 0} w(t - S_n).$$

(a) First, assume the points of $N$ form the renewal points $S_n$ of a renewal process with exponential interarrival times $Y_i \sim \exp(\alpha)$ ($i = 0, ...$), i.e., $S_n = Y_0 + ... + Y_n$. Derive the renewal equation for $\mathbb{E}[X(t)]$ and solve it.

Hint: recall the simple form of $\mathbb{E}[N(t)]$ here.

(b) Assume now that $S_n$ form the points of the PRM $N$, a homogeneous Poisson Point Process on $\mathbb{R}^+$ with intensity measure $\mu(dt) = \alpha dt$. Compute $\mathbb{E}[X(t)]$ in this case. Hint: Condition on knowing $N((0,t]) = k$ and use the order statistics property.

29. Let $X_i$ be i.i.d. Gaussian random variables with zero mean and variance 1. Let $\nu$ be a Poisson random variable with mean $m$. Describe the distribution of the point process $N(\cdot)$ given by

$$N(A) = \sum_{i=1}^{\nu} \mathbb{1}_{\{X_i \in A\}}$$

as fully as possible.

Hint: Set $\mu(A) = \mathbb{E}[N(A)]$. Find first $\mu(\mathbb{R})$, then $\mu(A)$ for arbitrary $A$. 