STAT 331 Homework 6

Due date: In class on Thursday, October 21, 2004

Instructor: Dr. Rudolf Riedi

26. (20 points)

    For the following non-negative functions compute the constant c which makes them a probability density. Then, compute their cumulative distribution functions, i.e., $F(a) = P[X \leq a]$) for *all* $a \in \mathbb{R}$. Also, compute their mean and variance, if they exist.

    (a) (6 points) $\mathcal{U}([0, 2\pi])$: Uniform on $[0, 2\pi]$

    $$f(x) = \begin{cases} c & \text{for } x \text{ in } [0, 2\pi], \\ 0 & \text{otherwise.} \end{cases}$$

    (b) (7 points) Cauchy:

    $$f(x) = \frac{c}{1 + x^2} \qquad \text{for every } x \text{ in } \mathbb{R}.$$

    ($c = 1/\pi$)

    (c) (7 points) $\exp(\lambda)$: One sided exponential with parameter $\lambda > 0$

    $$f(x) = \begin{cases} ce^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$

    ($c = \lambda$)

27. (10 points) Suppose $X \sim \mathcal{U}(-\frac{\pi}{2}, \frac{\pi}{2})$, i.e. X is a r.v. uniformly distributed between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Let $Y = \tan(X)$. The distribution of Y can be computed in two different ways.

    (a) Use $P[Y \leq y] = P[\tan(X) \leq y]$ to show that $F_Y(y) = \frac{1}{2} + \frac{1}{\pi}\arctan(y)$. Then infer the density $f_Y(y)$ of the distribution of Y as the derivative of $F_Y(y)$ and conclude that Y is a Cauchy r.v.

    (b) The following formula connecting the densities of the distributions of the random variables X and $Y = g(X)$ is valid if g is differentiable and strictly increasing:

    $$f_Y(g(x)) = \frac{f_X(x)}{g'(x)}, \qquad \text{or} \qquad f_Y(y) = f_X(g^{-1}(y)) \cdot (g^{-1})'(y).$$

    Use it to derive $f_Y(y)$ for $g(X) = \tan(X)$.

    This problem provides an intuitive interpretation of the Cauchy law: A person is blindfolded, standing in front of a wall. The person spins around a few times. The angle under which the person now “faces” the wall is completely arbitrary, in other words, uniformly distributed. The person starts walking straight. The location where the person will collide with the wall is random, with Cauchy distribution.

28. (10 points) Let the joint density of the pair of random variables (X, Y) be given by

    $$f_{XY}(x, y) = \begin{cases} y\exp(-xy) & \text{if } x > 1 \text{ and } y > 0 \\ 0 & \text{else.} \end{cases}$$

    (a) (4 points) Compute the marginal densities f_X and f_Y.

    (b) (2 points) Are X and Y independent? Show your reasoning.

    (c) (4 points) Compute $\mathbb{E}[X]$ and $\mathbb{E}[Y]$.