STAT 331 Homework 7

Due date: In class on Thursday, October 28, 2004

Instructor: Dr. Rudolf Riedi

Test 1 will be handed out October 28, due November 9.
All personal notes plus the hand-out lecture notes are allowed, as well as ONE book.
Handout solutions are NOT allowed.

Test 2 will be handed out November 23, due December 2.
Test 2 will be open books, notes and solutions.

Recall that a Gaussian (or Normal) random variable has the density

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

and characteristic function

$$\phi_X(u) = \exp\left(ju\mu - \frac{\sigma^2 u^2}{2}\right)$$

where $\mu = \mathbb{E}[X]$ and $\sigma^2 = \mathrm{var}(X)$. We use the notation $\mathcal{N}(\mu, \sigma^2)$ for such a distribution.

29. (10 points) Consider two independent Gaussian random variables: $X \sim \mathcal{N}(2, 1)$ and $Y \sim \mathcal{N}(1, 2^2)$. Set $Z = X - Y$.

    Find $\mathbb{E}[Z]$, $\mathrm{var}(Z)$, $P[Z \leq 0]$, $P[Y \leq 0]$, and $P[|Y| < 1/2]$. (1, 5, .327, .3085, .1747)

30. (10 points) Occasionally, a professor will drop the lowest quiz grade of each student. Clearly, this will increase the class average. We consider the specific case of two identical, independent quizzes where two lower of the two grades of each student is discarded. What is the resulting class in the following two cases?

    (a) The grades of both test are Uniform with mean μ and variance σ^2. ($\mu + \sigma/\sqrt{3}$)
    (b) The grades of both test are Normal with mean μ and variance σ^2. ($\mu + \sigma/\sqrt{\pi}$)

    Hint: Let X_k be the grade of quiz k. Thus, the probability that a given student received a grade, e.g., better than $\mu + \sigma$ in test 1 is $P[X_1 > \mu + \sigma]$. Let X denote the resulting grade of a student, i.e., $X = \max(X_1, X_2)$. Compute first the CDF of X, i.e., compute $P[X < a]$.

31. (10 points) A student takes a true/false test of 50 questions. Not having studied, the student answers all questions blindly without even looking at the question. There are no penalties for wrong answers, only points for correct answers.

    (a) (5 points:) What is the probability the student passes, meaning that he received at least 60% of the points? (.1013)
    (b) (5 points:) What is the approximative probability of passing using the Poisson distribution?

32. (10 points) A signal X sent through a channel, which adds noise N. Denote the detected signal by $Y = X + N$. Assume that X nad N are independent Gaussian r.v., $X \sim \mathcal{N}(0, 1)$ and $N \sim \mathcal{N}(0, \sigma^2)$.

    (a) (1 points) Given $X = a$ the random variable $Y|X = a$ is again Gaussian. What is its mean and variance? Give the density $f_{Y|X=a}(y)$.
    (b) (3 points) Use the law of total probability to compute the pdf of Y: f_Y. What is the distribution of Y? (Gaussian again, with mean... and variance...).
    (c) (3 points) Use Bayes' rule to compute the pdf of X given $Y = b$,i.e., compute $f_{X|Y=b}(a)$.
    (d) (3 points) Fix b. Find the a that maximizes $f_{X|Y=b}(a)$. ($a = b/(1 + \sigma^2)$)

    The relevance of this problem is as follows: detecting $Y = b$, one may estimate X to be equal to the a of (d). This is called a MAP estimator, of Maximum A Posteriori estimator, since $f_{X|Y=b}(a)$ is the posterior density of the unknown (to be estimated) X given the observation Y.