ELEC 535 Homework 1

Due date: In class on Friday, January 24, 2003

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Problem 1.1 (Coin flips)

A fair coin is flipped until the first head occurs. Let X denote the number of flips required. (a) Find the entropy $H(X)$ in bits. The following expressions may be useful:

$$\sum_{n=1}^{\infty} r^n = \frac{r}{1-r}, \qquad \sum_{n=1}^{\infty} n r^n = \frac{r}{(1-r)^2}.$$

(b) A random variable X is drawn according to this distribution. Find an "efficient" sequence of yes-no questions of the form, "Is X contained in the set S?" Compare $H(X)$ to the expected number of questions required to determine X.

Problem 1.2 (Entropy of functions of a random variable)

Let X be a discrete random variable. Show that the entropy of a function X is less than or equal to the entropy of X by justifying the following steps:

$$\begin{aligned} H(X, g(X)) &\overset{(a)}{=} H(X) + H(g(X)|X) \\ &\overset{(b)}{=} H(X); \\ H(X, g(X)) &\overset{(c)}{=} H(g(X)) + H(X|g(X)) \\ &\overset{(d)}{\geq} H(g(X)). \end{aligned}$$

Thus $H(g(X)) \leq H(X)$.

Problem 1.3 (Zero conditional entropy)

Show that if $H(Y|X) = 0$, then Y is a function of X, i.e., for all x with $p(x) > 0$, there is only one possible value of y with $p(x, y) > 0$.

Problem 1.4 (World Series)

The World Series is a seven-game series that terminates as soon as either team wins four games. Let X be the random variable that represents the outcome of a World Series between teams A and B; possible values of X are AAAA, BABABAB, and BBBAAAA, etc. Let Y be the number of games played, which range from 4 to 7. Assuming that A and B are equally matched and that the games are independent, calculate $H(X), H(Y), H(Y|X)$, and $H(X|Y)$.

Problem 1.5 (A metric)

A function $\rho(x, y)$ is a metric if for all x, y

- $\rho(x, y) \geq 0$
- $\rho(x, y) = \rho(y, x)$
- $\rho(x, y) = 0$ if and only if $x = y$
- $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$

(a) Show that $\rho(X, Y) = H(X|Y) + H(Y|X)$ satisfies the first, second, and fourth properties above. If we say that $X = Y$ if there is a one-to-one function mapping X to Y, then the third property is also satisfied, and $\rho(X, Y)$ is a metric.
(b) Verify that $\rho(X, Y)$ can also be expressed as

$$\begin{aligned}
\rho(X, Y) &= H(X) + H(Y) - 2I(X; Y) \\
&= H(X, Y) - I(X; Y) \\
&= 2H(X, Y) - H(X) - H(Y)
\end{aligned}$$