ELEC 535 Homework 3

Due date: In class on Friday, February 7, 2003

Instructor: Rudolf Riedi Rice University, Spring 2003

Problem 3.1 (Monotonic convergence of the empirical distribution)

Let $\widehat{p}_n$ denote the empirical probability mass function corresponding to $X_1, X_2, \ldots, X_n$ independent and identically distributed with distribution $p(x)$, $x \in \mathcal{X}$. Specifically,

$$\widehat{p}_n(x) = \frac{1}{n} \sum_{i=1}^{n} I(X_i = x)$$

is the proportion of times that $X_i = x$ in the first n samples, where I is an indicator function.
(a) Show for $\mathcal{X}$ binary that

$$ED(\widehat{p}_{2n}||p) \leq ED(\widehat{p}_n||p).$$

Thus the expected relative entropy "distance" from the empirical distribution to the true distribution decreases with sample size.
Hint: Write $\widehat{p}_{2n} = \frac{1}{2}\widehat{p}_n + \frac{1}{2}\widehat{p}'_n$ and use the convexity of D.
(b) Show for an arbitrary discrete $\mathcal{X}$ that

$$ED(\widehat{p}_n||p) \leq ED(\widehat{p}_{n-1}||p).$$

Problem 3.2 (Entropy of a disjoint mixture)

Let X_1 and X_2 be discrete random variables drawn according to probability mass functions p_1 over the set $\mathcal{X}_1 = \{1 \ldots m\}$ and p_2 over the set $\mathcal{X}_2 = \{m+1 \ldots n\}$, respectively.
Let X be a random variable which equals X_1 with probability α and X_2 with probability $1 - \alpha$.
(a) Find $H(X)$ in terms of $H(X_1)$, $H(X_2)$ and α.
(b) Maximize $H(X)$ over α to show that

$$2^{H(X)} \leq 2^{H(X_1)} + 2^{H(X_2)}$$

and interpret this result using the notion that $2^{H(X)}$ is the effective alphabet size.

Problem 3.3 (Measure of correlation)

Let X_1 and X_2 be identically distributed but not necessarily independent. Set

$$\rho = 1 - \frac{H(X_2|X_1)}{H(X_1)}$$

(a) Show that $\rho = I(X_1; X_2)/H(X_1)$.
(b) Show that $0 \leq \rho \leq 1$.
(c) When is $\rho = 0$?
(d) When is $\rho = 1$?