ELEC 535 Homework 6

Due date: In class on Friday, March 7, 2003

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Problem 6.1 (Simple optimum compression of a Markov source)

Consider the 3-state Markov process $U_1, U_2, \ldots$ having transition matrix

$U_{n-1} \backslash U_n$	S_1	S_2	S_3
S_1	1/2	1/4	1/4
S_2	1/4	1/2	1/4
S_3	0	1/2	1/2

Thus the probability that S_1 follows S_3 is equal to zero. Design 3 codes C_1, C_2, C_3 (one for each state S_1, S_2, S_3), each code mapping elements of the set of S_i's into sequences of 0's and 1's, such that this Markov process can be sent with maximal compression by the following scheme:

(a) Note the present symbol S_i.
(b) Select code C_i.
(c) Note the next symbol S_j and send the codeword in C_i corresponding to S_j.
(d) Repeat for the next symbol.

What is the average message length of the next symbol conditioned on the previous state $S = S_i$ using this coding scheme? What is the unconditional average number of bits per source symbol? Relate this to the entropy rate $H(\mathcal{U})$ of the Markov chain.

Problem 6.2 (Time's arrow)

Let $\{X_i\}_{i=-\infty}^{\infty}$ be a stationary stochastic process. Prove that

$$H(X_0|X_{-1}, X_{-2}, \ldots, X_{-n}) = H(X_0|X_1, X_2, \ldots, X_n).$$

In other words, the present has a conditional entropy given the past equal to the conditional entropy given the future.

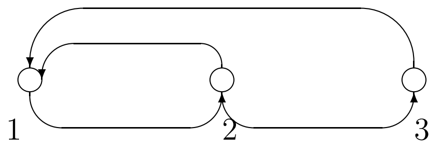
This is true even though it is quite easy to concoct stationary random processes for which the flow into the future looks quite different from the flow into the past. That is to say, one can determine the direction of time by looking at a sample function of the process. Nonetheless, given the present state, the conditional uncertainty of the next symbol in the future is equal to the conditional uncertainty of the previous symbol in the past.

Problem 6.3 (Entropy rate of constrained sequences.)

In magnetic recording, the mechanism of recording and reading the bits imposes constraints on the sequences of bits that can be recorded. For example, to ensure proper synchronization, it is often necessary to limit the length of runs of 0's between two 1's. Also to reduce inter-symbol interference, it may be necessary to require at least one 0 between any two 1's. We will consider a simple example of such a constraint.

Suppose that we are required to have at least one 0 and at most two 0's between any pair of 1's in a sequences. Thus, sequences like 101001 and 0101001 are valid sequences, but 0110010 and 0000101 are not. We wish to calculate the number of valid sequences of length n.

(a) Show that the set of constrained sequences is the same as the set of allowed paths on the following state diagram.



ß(b) Let $X_i(n)$ be the number of valid paths of length n ending at state i. Argue that $\mathbf{X}(n) = [X_1(n)\, X_2(n)\, X_3(n)]^T$ satisfies the following recursion:

$$\begin{bmatrix} X_1(n) \\ X_2(n) \\ X_3(n) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} X_1(n-1) \\ X_2(n-1) \\ X_3(n-1) \end{bmatrix} = A\mathbf{X}(n-1)$$

with initial condition $\mathbf{X}(1) = [1\,1\,0]^T$.

(c) Then we have by induction

$$\mathbf{X}(n) = A\mathbf{X}(n-1) = A^2\mathbf{X}(n-2) = \cdots = A^{n-1}\mathbf{X}(1).$$

Using the eigenvalue decomposition of A for the case of distinct eigenvalues, we can write $A = U^{-1}\Lambda U$, where Λ is the diagonal matrix of eigenvalues. Then $A^{n-1} = U^{-1}\Lambda^{n-1}U$. Show that we can write

$$\mathbf{X}(n) = \lambda_1^{n-1}\mathbf{Y}_1 + \lambda_2^{n-1}\mathbf{Y}_2 + \lambda_3^{n-1}\mathbf{Y}_3,$$

where $\mathbf{Y}_1, \mathbf{Y}_2, \mathbf{Y}_3$ do not depend on n. For large n, this sum is dominated by the largest term. Therefore argue that for $i = 1, 2, 3$, we have

$$\frac{1}{n}\log X_i(n) \to \log\lambda,$$

where λ is the largest (positive) engenvalues. Thus the number of sequences of length n grows as λ^n for large n. Calculate λ for the matrix A above. (The case when the eigenvalues are not distinct can be handled in a similar manner.)

ß(d) We will now take a different approach. Consider a Markov chain whose state diagram is the one given in part (a), but with arbitrary transition probabilities. Therefore the probability transition matrix of this Markov chain is

$$P = \begin{bmatrix} 0 & \alpha & 1 \\ 1 & 0 & 0 \\ 0 & 1-\alpha & 0 \end{bmatrix}.$$

Show that the stationary distribution of the Markov chain is

$$\mu = \left[\frac{1}{3-\alpha}, \frac{1}{3-\alpha}, \frac{1-\alpha}{3-\alpha}\right]^T.$$

(e) Maximize the entropy rate of the Markov chain over choices of α. What is the maximum entropy rate of the chain?

(f) Compare the maximum entropy rate in part (e) with $\log\lambda$ in part (c). Why are the two answers the same?