ELEC 535 Homework 8

Due date: In class on Monday, April 7, 2003

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Problem 8.1 (Channel capacity.)

Consider the discrete memoryless channel $Y = X + Z$ (mod) 11, where

$$Z = \begin{pmatrix} 1, & 2, & 3 \\ 1/3, & 1/3, & 1/3 \end{pmatrix}$$

and $X \in \{0, 1, \ldots, 10\}$. Assume that Z is independent of X.

1. Find the capacity.
2. What is the maximizing $p^*(x)$?

Problem 8.2 (Zero-error capacity.)

A channel with alphabet $\{0, 1, 2, 3, 4\}$ has transmission probabilities of the form

$$p(y|x) = \begin{cases} 1/2 & \text{if } y = x \pm 1 \bmod 5 \\ 0 & \text{otherwise} \end{cases} .$$

1. Compute the capacity of this channel in bits.
2. The zero-error capacity of a channel is the number of bits per channel use that can be transmitted with zero probability of error. Clearly, the zero-error capacity of this pentagonal channel is at least 1 bit (transmit 0 or 1 with probability 1/2). Find a block code that shows that the zero-error capacity is greater than 1 bit. Can you estimate the exact value of the zero-error capacity?
(Hint: Consider codes of length 2 for this channel.)

Problem 8.3 (Maximum likelihood decoding.)

A source produces independent, equally probable symbols from an alphabet (a_1, a_2) at a rate of one symbol every 3 seconds. These symbols are transmitted over a binary symmetric channel which is used once each second by encoding the source symbol a_1 as 000 and the source symbol a_2 as 111. If in the corresponding 3 second interval of the channel output, any of the sequences 000,001,010,100 is received, a_1 is decoded; otherwise a_2 is decoded. Let $\epsilon < 1/2$ be the channel crossover probability.

1. For each possible received 3-bit sequence in the interval corresponding to a given source letter, find the probability that a_1 came out of the source given that received sequence.
Hint: Recall Bayes' rule: for any events A and B,

$$\Pr(A|B) = \frac{\Pr(A)\Pr(B|A)}{\Pr(B)}$$

2. Using the above part (1), show that the above decoding rule minimizes the probability of an incorrect decision.
3. Find the probability of an incorrect decision (using part (1) is not the easy way here).
4. If the source is slowed down to produce one letter every $2n+1$ seconds, a_1 being encoded by $2n+1$ 0's and a_2 being encoded by $2n+1$ 1's. What decision rule minimizes the probability of error at the decoder? Find the probability of error as $n \to \infty$.

Problem 8.4 (Channels with memory have higher capacity.)

Consider a binary symmetric channel with $Y_i = X_i \oplus Z_i$ where $\oplus$ is mod 2 addition, and $X_i, Y_i \in \{0, 1\}$. Suppose that $\{Z_i\}$ has constant marginal probabilities $\Pr\{Z_i = 1\} = p = 1 - \Pr\{Z_i = 0\}$, but that $Z_1, Z_2, \ldots, Z_n$ are not necessarily independent. Assume that Z^n is independent of the input X^n. Let $C = 1 - H(p, 1-p)$. Show that $\max_{p(x_1,x_2,\ldots,x_n)} I(X_1, X_2, \ldots, X_n; Y_1, Y_2, \ldots, Y_n) \geq nC$.
(Hint: Use that $X^n|Y^n = Z^n|Y^n$ and express $I(X^n; Y^n)$ in terms of entropies.)

Problem 8.5 (Differential entropy.)

Evaluate the differential entropy $h(X) = -\int f \ln f$ for the following:

1. The exponential density, $f(x) = \lambda e^{-\lambda x}, x \geq 0$.
2. The Laplacian density, $f(x) = \frac{1}{2}\lambda e^{-\lambda |x|}$.
3. The sum of X_1 and X_2, where X_1 and X_2 are independent normal random variables with means μ_i and variances σ_i^2, $i = 1, 2$.