STAT 582 Homework 1

Due date: In class on Friday, February 4, 2005

Instructor: Dr. Rudolf Riedi

1. Let X_n be a monotone sequence of random variables. Assume that $X_n \stackrel{P}{\to} X$. Show that $X_n \stackrel{\text{a.s.}}{\to} X$.
   Hint: Consider subsequences.

2. Let X_n be any sequence of random variables. Show that

$$X_n \stackrel{\text{a.s.}}{\to} X \qquad \Longleftrightarrow \qquad \sup_{k \geq n} |X_k - X| \stackrel{P}{\to} 0.$$

   Hint: this is very similar to a result developed in class.

3. Give an example of a probability space such that for all $0 < p < 1$ the expression $||X||_p = (\mathbb{E}|X|^p)^{1/p}$ is not a norm, i.e., such that the triangular inequality $||X + Y||_p \leq ||X||_p + ||Y||_p$ fails for some random variables X and Y.
   Hint: Consider a simple coin toss, and set X and Y such that $X + Y = 1$ always.

4. Let X_n be any sequence of random variables. Set $S_n = X_1 + \ldots X_n$.
   (a) Assume that the sequence of real numbers a_n converges to a. Show that then also $(a_1 + \ldots + a_n)/n$ converges to a.
   (b) Conclude that if $X_n \stackrel{\text{a.s.}}{\to} 0$ then $S_n/n \stackrel{\text{a.s.}}{\to} 0$.
   (c) Let $p \geq 1$.Show that if $X_n \stackrel{L_p}{\to} 0$ then $S_n/n \stackrel{L_p}{\to} 0$. Hint: triangular inequality.
   (d) Show that $X_n \stackrel{P}{\to} 0$ does NOT imply $S_n/n \stackrel{P}{\to} 0$.
   Hint: Consider a suitable sequence of random variables with $X_n = 2^n$ with probability $1/n$, and $X_n = 0$ else.