# STAT 582 Homework 6

Due date: In class on Friday, March 25, 2005

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17. Let $X_n$ be a sequence of random variables such that

$$P[X_n = n] = 1/n \qquad P[X_n = 0] = 1 - 1/n.$$

   (a) Does the sequence $\{X_n\}_n$ converge in probability? If so to what limit? Show your argument.

   (b) Does the sequence $\{X_n\}_n$ converge in distribution? If so to what limit? Show your argument.

   (c) Assume in addition that the random variables in the sequence $\{X_n\}_n$ are mutually independent. Does the sequence $\{X_n\}_n$ converge almost surely? If so to what limit? Show your argument. Hint: compute first what $\limsup_n X_n$ and $\liminf_n X_n$ are almost surely.

18. Suppose $U_n$ ($n \in \mathbb{N}$) are i.i.d. random variables, uniformly distributed on $[0, 1]$. Set

$$Z_n := (U_1 \cdots U_n)^{1/n} = \prod_{k=1}^{n} U_k^{1/n}.$$

Show that $Z_n$ converges almost surely. Hint: SLLN

19. Let $X_n$ ($n \in \mathbb{N}$) be non-negative, integer valued random variables. Let $F_n$ be the (cumulative) distribution function of $X_n$, i.e., $F_n(x) = P[X_n \leq x]$.

   (a) Show that $F_n$ converges weakly if and only if for every non-negative integer $k$ we have

$$P[X_n = k] \to p_k \qquad \text{as } n \to \infty$$

   where $\{p_k\}_k$ is some sequence of non-negative numbers. Provided that it exists, express the weak limit $F$ of the sequence $\{F_n\}_n$ in terms of the $p_k$.
   Hint: Explain first why it is enough to study $F_n(x)$ (as $n \to \infty$) for *non-integer* $x$.

   (b) Conclude that weak and vague convergence are equivalent for non-negative integer valued random variables (but not necessary with the same limit). To clarify whether this property is due to the integer values or due to the non-negative values do the following: Show that the distributions of $Y_n = (-1)^n n$ converges vaguely but not weakly. (the limiting distribution can not be proper; what is it?) Show that if $G_n$ converges vaguely to $G$ and there exists $a^*$ such that $G_n(a^*) = 0$ for all $n$ then $G_n$ converges also weakly to $G'$ for some $G'$. Is $G$ necessarily proper?

   (c) Assume that $F_n$ converges weakly to $F$. Show that necessarily $p_0 + \ldots + p_m \leq 1$ for all $m$. Conclude that $\sum_k p_k \leq 1$. Under what conditions does $F_n$ converge in distribution (meaning: under what conditions is $F$ proper).

The following is an application of the above.

20. Let $N_k$ be a sequence of Poisson random variables of mean $\lambda_k$, i.e., $P[N_k = m] = e^{-\lambda_k} \frac{(\lambda_k)^m}{m!}$ for integer $m \geq 0$.

   (a) Show that the distribution functions of $N_k$ converge weakly iff $\lambda_k \to \lambda$ where $0 \leq \lambda \leq \infty$. (Recall that $\lambda_n \to \infty$ means that for all $M > 0$ there exists $n_0$ such that $\lambda_n \geq M$ for all $n > n_0$. Sometimes such a sequence is said to "diverge to (plus) infinity". Usually —and certainly in this course— such a sequence is not considered "convergent".)

   (b) Show that the distribution functions of $N_k$ converge properly (or equivalently: $N_k$ converges in distribution) iff $\lambda_k$ converges, i.e, $\lambda_k \to \lambda$ where $0 \leq \lambda < \infty$. Is the limiting distribution Poisson again? If so, with what mean?

   (c) What is the limiting df if $\lambda_k \to \infty$? Hint: this can not be a proper df.

   (d) Let $N$ be Poisson with parameter $\lambda$. Verify by direct computation that $\mathbb{E}[N] = \lambda$ and $\text{var}(N) = \lambda$. Hint: to compute the mean use that $k/k! = 1/(k-1)!$ and $\sum_{k=0}^{\infty} \lambda^k/k! = e^{\lambda}$; to compute second order moments, start by computing $\mathbb{E}[N(N-1)]$ and use the earlier hint.