STAT 582 Homework 7

Due date: In class on Wednesday, April 13, 2005

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21. Let X_n be a sequence of Gaussian random variables with means μ_n and variances σ_n^2. Assume that $X_n \xrightarrow{\mathrm{D}} X$. Show that necessarily $\mu_n \to \mu$ for some $\mu \in \mathbb{R}$ and $\sigma_n^2 \to \sigma^2$ for some $\sigma^2 \geq 0$. Hint: Continuity theorem.

22. Let N_k be a sequence of independent Poisson random variables of mean $\lambda_n > 0$, i.e., $P[N_k = m] = \exp(-\lambda_k)\frac{(\lambda_k)^m}{m!}$ for integer $m \geq 0$.

    (a) Show that the characteristic function ϕ_n of N_n is $\phi_n(t) = \exp(\lambda_n(e^{it} - 1))$.

    (b) Conclude that the sum $S_m = N_1 + \ldots N_m$ is again Poisson. Hint: Uniqueness theorem.

    (c) Show that $\sum_k N_k$ converges in distribution iff $\sum_n \lambda_n < \infty$ using the continuity theorem.

23. Let B_k be a sequence of independent Bernoulli random variables with $P[B_k = 1] = 1 - P[B_k = 0] = p_k$. Let $M_n = B_1 + \ldots + B_n$ be the associated Binomial variable.

    (a) Show that the characteristic function of M_n is $\phi_n(t) = \prod_{k=1}^{n}(1 + p_k(e^{it} - 1))$.

    (b) Fix $\lambda > 0$. For every given n define M_n as above with $p_k = \lambda/n$ $(k = 1, .., n)$. Now let $n \to \infty$. Show that $M_n \xrightarrow{\mathrm{D}} Y$ for some Poisson r.v. Y. What is the mean of Y?

    (c) Assume that $p_n = 1/2$ for all n. Let $X_n = 2B_n - 1$. Show that the characteristic function of X_n is $\cos(t)$. Let $S_n = X_1 + \ldots X_n = 2M_n - n$. Show that $S_n/\sqrt{n} = (2M_n - n)/\sqrt{n}$ converges in distribution to a standard normal; do so from "scratch" using the continuity theorem, but without using the CLT. Hint: the characteristic function of S_n is $\cos^n(t/\sqrt{n})$; approximate $\cos(u) = 1 - u^2/2$.

    (d) Why are (b) and (c) not in violation of the convergence to types theorem? (After all, (b) and (c) provide different distributional limits of renormalizations of M_n.)