Let $U$ be a discrete random variable with values $\{u_i\}_{i \in I}$. Let $A_i := \{\omega : U(\omega) = u_i\}$.

Let $Y$ be in $L^1$. Recall that
$$\mathbb{E}[Y | \sigma(A_i, i \in I)] = \mathbb{E}[Y | \sigma(C_i, i \in I)] = \sum_{i \in I} a_i \mathbb{I}_{A_i}(\omega)$$

(1)

where the $a_i$ must satisfy the condition $\mathbb{E}[a_i \mathbb{I}_{A_i}] = \mathbb{E}[Y \mathbb{I}_{A_i}]$ and can otherwise be chosen arbitrarily.

Verify the following basic properties of conditional expectations for discrete random variables by explicit computation using (1).

(a) Show that $\mathbb{E}[Y | \{\emptyset, \Omega\}] = \mathbb{E}[Y | \{\emptyset, \Omega\}] = \mathbb{E}[Y]$. Show that $Y$ is independent of $\{\emptyset, \Omega\}$.

(b) More generally, assume that $Y$ is independent of $U$. Show that $\mathbb{E}[Y | U] = \mathbb{E}[Y]$.

(c) Let $h(x, y)$ be measurable and assume that $h(U, Y)$ lies in $L^1$. Argue that
$$\mathbb{E}[h(U, Y) | U] = \sum_{i \in I} \alpha_i \mathbb{I}_{A_i}(\omega)$$

and compute $\alpha_i$. Conclude the following two facts.

- If $Y$ and $U$ are independent, then we have $\alpha_i P(A_i) = \mathbb{E}[h(u_i, Y)] P(A_i)$ for all $i$. In other words,
$$\mathbb{E}[h(U, Y) | U] = f(U)$$
where $f(x) = \mathbb{E}[h(x, Y)]$.

- If $g$ is measurable such that $g(U)Y$ is in $L^1$ then
$$\mathbb{E}[g(U)Y | U] = g(U)\mathbb{E}[Y | U]$$


(a) Compute $\mathbb{E}[S[U]]$ and $\mathbb{E}[S[V]]$.

(b) Compute $\mathbb{E}[S[U, V]]$.

(c) Compute $\mathbb{E}[U[S]]$.

In case of doubt note that $\mathbb{E}[S[U, V]] := \mathbb{E}[S[U, V]]$ and setting $C_{i,j} = \{\omega : U(\omega) = u_i, V(\omega) = v_j\}$ we have $\sigma(U, V) = \sigma(C_{i,j} : i \in I, j \in J) = \{\cup_{(i,j) \in K} C_{i,j} : K \subset I \times J\}$. In this notation we have
$$\mathbb{E}[S[U, V]] = \sum_{(i,j) \in I \times J} c_{i,j} \mathbb{I}_{C_{i,j}}$$
26. A sequence of random variables \( \{U_n\}_n \) is called a \textit{martingale} with respect to \( \mathcal{F}_n \) iff for each \( n \),

(a) \( \mathcal{F}_n \) is a subfield of \( \mathcal{F} \) and \( \mathcal{F}_n \subset \mathcal{F}_{n+1} \),
(b) \( U_n \) is in \( L^1 \) and \( \mathcal{F}_n \)-measurable, and
(c) \( \mathbb{E}[U_{n+1}|\mathcal{F}_n] = U_n \).

Let \( X_n \) be iid zero mean random variables, and let \( Y_n \) be iid mean 1 random variables. Let \( S_n = X_1 + \ldots + X_n \), let \( T_n = Y_1 \ldots Y_n \) and assume that \( S_n \) and \( T_n \) are in \( L^1 \) for all \( n \).

(a) Assume that \( U_n \) is a martingale. Show that \( \mathbb{E}[U_{n+1}] = \mathbb{E}[U_n] = \mathbb{E}[U_1] \). Hint: use property (c) of a martingale.

(b) Show, that \( S_n \) is a martingale with respect to \( \mathcal{F}_n := \sigma(X_1, \ldots, X_n) \).

(c) What is wrong about the following statement (consult the properties (a)-(c) of a martingale): “\( S_n \) is a martingale with respect to \( \mathcal{H}_n := \sigma(S_n) \).”

(d) Show that \( T_n \) is a martingale with respect to \( \mathcal{G}_n := \sigma(Y_1, \ldots, Y_n) \).

(e) Martingale compensator. Assume that \( P[X_n = 1] = P[X_n = -1] = 1/2 \); so, \( S_n \) is a random walk.

i. Use (a) to show that if the sequence \( a_n \) is such that \( U_n := S_n^2 - a_n \) is a martingale with respect to \( \mathcal{F}_n \), then necessarily \( a_n = n \).

ii. Bonus: Show that \( S_n^2 - n \) is indeed a martingale. The sequence \( a_n \) is called martingale compensator.