STAT 582 Homework 8

Due date: In class on Monday, April 25, 2005

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24. Let U be a discrete random variable with values $\{u_i\}_{i\in I}$. Let $A_i := \{\omega : U(\omega) = u_i\}$.
Let Y be in L^1. Recall that

$$\mathbb{E}[Y|U] = \mathbb{E}[Y|\ \sigma(A_i, i \in I)] = \sum_{i\in I} a_i \mathbb{I}_{A_i}(\omega) \tag{1}$$

where the a_i must satisfy the condition $\mathbb{E}[a_i \mathbb{I}_{A_i}] = \mathbb{E}[Y \mathbb{I}_{A_i}]$ and can otherwise be chosen arbitrarily.
Verify the following basic properties of conditional expectations for discrete random variables by explicit computation using (1).

(a) Show that $\mathbb{E}[Y|\mathbb{I}_\Omega] = \mathbb{E}[Y|\{\emptyset, \Omega\}] = \mathbb{E}[Y]$. Show that Y is independent of $\mathbb{I}_\Omega$.

(b) More generally, assume that Y is independent of U. Show that $\mathbb{E}[Y|U] = \mathbb{E}[Y]$.

(c) Let $h(x, y)$ be measurable and assume that $h(U, Y)$ lies in L^1. Argue that

$$\mathbb{E}[h(U, Y)|U] = \sum_{i\in I} \alpha_i \mathbb{I}_{A_i}(\omega)$$

and compute α_i. Conclude the following two facts.

- If Y and U are independent, then we have $\alpha_i P(A_i) = \mathbb{E}[h(u_i, Y)]P(A_i)$ for all i. In other words,
$$\mathbb{E}[h(U, Y)|U] = f(U) \qquad \text{where } f(x) = \mathbb{E}[h(x, Y)].$$
- If g is measurable such that $g(U)Y$ is in L^1 then
$$\mathbb{E}[g(U)Y|U] = g(U)\mathbb{E}[Y|U]$$

25. Let U and V be two i.i.d. Bernoulli random variables with $P[U = 0] = P[U = 1] = P[V = 0] = P[V = 1] = 1/2$ (toss two fair coins). Let $S = U + V$.

(a) Compute $\mathbb{E}[S|U]$ and $\mathbb{E}[S|V]$.

(b) Compute $\mathbb{E}[S|U, V]$.

(c) Compute $\mathbb{E}[U|S]$.

In case of doubt note that $\mathbb{E}[S|U, V] := \mathbb{E}[S|\sigma(U, V)]$ and setting $C_{i,j} = \{\omega : U(\omega) = u_i, V(\omega) = v_j\}$ we have $\sigma(U, V) = \sigma(C_{i,j} : i \in I, j \in J) = \{\cup_{(i,j)\in K} C_{i,j} \quad K \subset I \times J\}$. In this notation we have

$$\mathbb{E}[S|U, V] = \sum_{(i,j)\in I\times J} c_{i,j} \mathbb{I}_{C_{i,j}}$$

26. A sequence of random variables $\{U_n\}_n$ is called a *martingale* with respect to $\mathcal{F}_n$ iff for each n,

(a) $\mathcal{F}_n$ is a subfield of $\mathcal{F}$ and $\mathcal{F}_n \subset \mathcal{F}_{n+1}$,

(b) U_n is in L^1 and $\mathcal{F}_n$-measurable, and

(c) $\mathbb{E}[U_{n+1}|\mathcal{F}_n] = U_n$.

Let X_n be iid zero mean random variables, and let Y_n be iid mean 1 random variables. Let $S_n = X_1 + \ldots + X_n$, let $T_n = Y_1 \ldots Y_n$ and assume that S_n and T_n are in L^1 for all n.

(a) Assume that U_n is a martingale. Show that $\mathbb{E}[U_{n+1}] = \mathbb{E}[U_n] = \mathbb{E}[U_1]$. Hint: use property (c) of a martingale.

(b) Show, that S_n is a martingale with respect to $\mathcal{F}_n := \sigma(X_1, \ldots, X_n)$.

(c) What is wrong about the following statement (consult the properties (a)-(c) of a martingale): "S_n is a martingale with respect to $\mathcal{H}_n := \sigma(S_n)$."

(d) Show that T_n is a martingale with respect to $\mathcal{G}_n := \sigma(Y_1, \ldots, Y_n)$.

(e) Martingale compensator. Assume that $P[X_n = 1] = P[X_n = -1] = 1/2$; so, S_n is a random walk.

   i. Use (a) to show that if the sequence a_n is such that $U_n := S_n^2 - a_n$ is a martingale with respect to $\mathcal{F}_n$, then necessarily $a_n = n$.

   ii. Bonus: Show that $S_n^2 - n$ is indeed a martingale. The sequence a_n is called martingale compensator.