1. (a) Show the following fact.

Let \( \{a_n\}_n \) be a non-decreasing sequence of real numbers. If any subsequence \( \{a_{n(k)}\}_k \) converges, then the entire series \( \{a_n\}_n \) converges.

Hint: Proof that \( \{a_n\}_n \) converges if and only if it is bounded which occurs if and only if any subsequence is bounded. For both parts, you will need to use the monotonicity!

(b) Let \( X_n \) be a non-decreasing sequence of random variables. Assume that \( X_n \xrightarrow{P} X \). Show that \( X_n \xrightarrow{a.s.} X \).

Hint: Use (a).

2. Let \( X_n \) be a sequence of r.v. Assume that there exists a measurable set \( N \) with \( P[N] = 0 \) with the following property: for every subsequence \( \{n(k)\}_k \) there exists a sub-subsequence \( \{n(k(i))\}_i \) such that

\[
X_{n(k(i))}(\omega) \to X(\omega)
\]

for each \( \omega \notin N \). Conclude that \( X_n \xrightarrow{a.s.} X \).

3. Give an example of a probability space such that for all \( 0 < p < 1 \) the expression \( \|X\|_p = (\mathbb{E}|X|^p)^{1/p} \) is not a norm, i.e., such that the triangular inequality \( \|X + Y\|_p \leq \|X\|_p + \|Y\|_p \) fails for some random variables \( X \) and \( Y \).

Hint: Consider a simple coin toss, and set \( X \) and \( Y \) such that \( X + Y = 1 \) always.

4. Let \( X_n \) be any sequence of random variables. Set \( S_n = X_1 + \ldots + X_n \).

(a) Assume that the sequence of real numbers \( a_n \) converges to \( a \). Show that then also \( (a_1 + \ldots + a_n)/n \) converges to \( a \).

(b) Conclude that if \( X_n \xrightarrow{a.s.} 0 \) then \( S_n/n \xrightarrow{a.s.} 0 \).

(c) Let \( p \geq 1 \). Show that if \( X_n \xrightarrow{L_p} 0 \) then \( S_n/n \xrightarrow{L_p} 0 \). Hint: triangular inequality.

(d) Show that \( X_n \xrightarrow{P} 0 \) does NOT imply \( S_n/n \xrightarrow{P} 0 \).

Hint: Consider a suitable sequence of random variables with \( X_n = 2^n \) with probability \( 1/n \), and \( X_n = 0 \) else.