STAT 582 Homework 1

Due date: In class on Friday, February 3, 2006

Instructor: Dr. Rudolf Riedi

1. (a) Show the following fact.

   Let $\{a_n\}_n$ be a non-decreasing sequence of real numbers. If any subsequence $\{a_{n(k)}\}_k$ converges, then the entire series $\{a_n\}_n$ converges.

   Hint: Proof that $\{a_n\}_n$ converges if and only if it is bounded which occurs if and only if any subsequence is bounded. For both parts, you will need to use the monotonicity!

   (b) Let X_n be a non-decreasing sequence of random variables. Assume that $X_n \overset{P}{\to} X$. Show that $X_n \overset{\text{a.s.}}{\to} X$.

   Hint: Use (a).

2. Let X_n be a sequence of r.v. Assume that there exists a measurable set N with $P[N] = 0$ with the following property: for every subsequence $\{n(k)\}_k$ there exists a sub-subsequence $\{n(k(i))\}_i$ such that

$$X_{n(k(i))}(\omega) \to X(\omega)$$

   for each $\omega \notin N$. Conclude that $X_n \overset{\text{a.s.}}{\to} X$.

3. Give an example of a probability space such that for all $0 < p < 1$ the expression $||X||_p = (\mathbb{E}|X|^p)^{1/p}$ is not a norm, i.e., such that the triangular inequality $||X + Y||_p \leq ||X||_p + ||Y||_p$ fails for some random variables X and Y.
   Hint: Consider a simple coin toss, and set X and Y such that $X + Y = 1$ always.

4. Let X_n be any sequence of random variables. Set $S_n = X_1 + \ldots X_n$.

   (a) Assume that the sequence of real numbers a_n converges to a. Show that then also $(a_1 + \ldots + a_n)/n$ converges to a.

   (b) Conclude that if $X_n \overset{\text{a.s.}}{\to} 0$ then $S_n/n \overset{\text{a.s.}}{\to} 0$.

   (c) Let $p \geq 1$.Show that if $X_n \overset{L_p}{\to} 0$ then $S_n/n \overset{L_p}{\to} 0$. Hint: triangular inequality.

   (d) Show that $X_n \overset{P}{\to} 0$ does NOT imply $S_n/n \overset{P}{\to} 0$.
   Hint: Consider a suitable sequence of random variables with $X_n = 2^n$ with probability $1/n$, and $X_n = 0$ else.