5. Let $X_n$ have normal distribution with mean 0 and variance $\sigma_n^2$.

   (a) Assume that $\{X_n\}$ is u.i. Show that the variances must be uniformly bounded, i.e., there exists $K$ such that $\sigma_n \leq K$ for all $n$.

   (b) Assume that the variances are uniformly bounded, i.e., assume there exists $K$ such that $\sigma_n \leq K$ for all $n$. Show that for all $n$

\[
\int_{|X_n|>a} |X_n| dP \leq \frac{2}{\sqrt{2\pi}} K \int_{a/K}^{\infty} y \exp\left(-\frac{y^2}{2}\right) dy
\]

   and conclude that $\{X_n\}$ is u.i.

   In summary, you showed that a sequence of zero mean normal variables is u.i. iff their variances are uniformly bounded.

6. Suppose $\{X_n\}$ and $\{Y_n\}$ are two u.i. families defined on the same probability space. Is $\{X_n + Y_n\}$ u.i.? Show your argument.

   Hint: Triangular inequality $|X_n + Y_n| \leq |X_n| + |Y_n|$ a.s.

7. Let $\{X_n\}$ be a sequence of i.i.d. random variables with mean zero and variance $\sigma^2$. Let $a_n$ be a sequence of real numbers. Set

\[
S_n = \sum_{i=1}^{n} a_i X_i.
\]

   Prove: $\{S_n\}$ converges in $L_2 \iff s_n := \sum_{i=1}^{n} a_i^2$ converges in $\mathbb{R}$.

   Hint: you can not use the limiting random variable $\sum_{i=1}^{\infty} a_i X_i$ to show convergence before you have not established that the limit actually exists. So, use the C... criterium.

8. Let $\{Y_n\}$ be a sequence of independent Gaussian random variables with mean zero and variance $\sigma_n^2$. Set

\[
S_n = \sum_{i=1}^{n} a_i X_i.
\]

   Under what assumptions on the sequence of variances does $S_n$ converge in $L_2$?