STAT 582 Homework 3

Due date: In class on Wednesday, March 1, 2006

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9. [A Kinchine-type L_1-LLN. This could be an exam question.] Let X_n be iid in L_1. Let $\bar{X}_n = (1/n)\sum_{k=1...n} X_k$.
   (a) Show that $\bar{X}_n$ has uniformly bounded first moments.
   Hint: Triangular inequality.
   (b) Show that $\bar{X}_n$ is uniformly absolutely continuous.
   Hint: Use that $Q(A) = \int_A |X_1| dP$ is absolutely continuous and again the triangular inequality.
   (c) Conclude that $\bar{X}_n$ converges in L_1. Hint: use theorems from class. no "new proofs" needed.

10. Here, we establish a simplified version of the Three Series Theorem for positive random variables, sometimes called Two-Series Theorem.

    Assume that $\{X_n\}$ are positive. Show that

    if $\sum_n \mathbb{E}[X_n \mathbb{I}_{\{|X_n|<c\}}]$ converges,
    then $\sum_n \mathrm{var}(X_n \mathbb{I}_{\{|X_n|<c\}}) < \infty$.

    In conclusion (Two-Series Theorem): The sum of positive independent r.v. converges almost surely iff the two series (i) and (iii) of the Three Series Theorem converge.

    Hint: Recall that the variance is bounded by the second moment.

11. Let $\{Z_n\}_n$ be a sequence of exponential random variables with mean one, i.e., $P[Z_n > x] = \exp(-x)$. Let $\{\lambda_n\}_n$ be a sequence of strictly positive numbers. Set $X_n = Z_n/\lambda_n$; then, obviously, X_n is exponential with $P[X_n > x] = \exp(-x\lambda_n)$.
    (a) Assume that X_n is u.i. Conclude that there exists a constant $\theta > 0$ such that $\lambda_n \geq \theta$ for all n; we say that "the sequence λ_n is bounded away from zero".
    (b) Vice versa, assume that the sequence λ_n is bounded away from zero. Show that X_n is then u.i.
    Hint: Use that the Z_n have identical distribution.
    (c) Assume that X_n converges in L_p for some $p \geq 1$. Show that the sequence λ_n converges to a positive, non-zero number or diverges to ∞.
    Hint: Use moments.
    (d) Assume that Y_n are non-negative random variables. Show that $T_n = Y_1 + ... + Y_n$ converges in L_1 iff $\sum_n \mathbb{E}[Y_n] < \infty$. Conclude that for exponential variables X_n as above we have that $\sum_n X_n$ converges in L_1 iff $\sum_n 1/\lambda_n < \infty$.
    Hint: Cauchy criterium as for the earlier similar problem about L_2.

    Note: In this example you can verify explicitly that convergence in L_1 implies u.i. (Indeed, if the strictly positive sequence λ_n goes to a positive, non-zero number or diverges to ∞ then certainly it is bounded away from zero.)

12. Let $\{Z_n\}_n$ be a sequence of *independent* exponential random variables with mean one, i.e., $P[Z_n > x] = \exp(-x)$. Let $\{\lambda_n\}_n$ be a sequence of strictly positive numbers. Set $X_n = Z_n/\lambda_n$ and $S_n = X_1 + ... + X_n$.
    (a) Let $c > 0$. Show that

    $$\mathbb{E}[X_n \mathbb{I}_{\{X_n<c\}}] = \frac{1}{\lambda_n} - \left(c + \frac{1}{\lambda_n}\right) e^{-c\lambda_n} \tag{1}$$

    (b) Assume that S_n converges a.s. Show that necessarily $\sum_n 1/\lambda_n < \infty$.

(c) Assume now that $\sum_n 1/\lambda_n < \infty$. Show that S_n converges almost surely.

Hints:

(a) Direct computation.

(b) Proceed along the following steps.
(i) Use that $P[X_n > x] = \exp(-x\lambda_n)$ to show $\sum_n \exp(-c\lambda_n) < \infty$.
(ii) Conclude that $\lambda_n \to \infty$.
(iii) Using this fact conclude that the second additive terms in (1) can be summed over n. Conclude that the first additive terms can be summed, which is the claim.

(c) You may proceed similarly as before in (b) by concluding first that $\lambda_n \to \infty$ and then applying the Two-Series-Theorem.
Alternatively, and maybe even faster, you can apply Kolmogorov's convergence criterium by which you even get convergence in L_2 for free.
Finally, a third (and quickest) possibility is to combine one of the results developed in this set of questions with theorems from class.

Note the following fact about sequences of *independent* random variables X_n:

[α] Either $X_n \stackrel{\text{a.s.}}{\to} Z$ or $P[\{\omega : X_n(\omega) \text{ is not Cauchy in } \mathbb{R}\}] = 1$.

[β] If $X_n \stackrel{\text{a.s.}}{\to} Z$ for some random variable Z then there exists a constant c such that $Z(\omega) = c$ a.s.

[γ] If $X_n \stackrel{P}{\to} Z$ for some random variable Z then there exists a constant c such that $Z(\omega) = c$ a.s.

[δ] If X_n are i.i.d. then, almost surely, $X_n(\omega)$ does not converge and X_n does not converge in probability — unless there exists c such that $X_n = c$ almost surely, for all n.

The proof uses Kolmogorov zero-one law in [α] and [β]. In [γ] use that there is a subsequence that converges a.s. and so Z has to be a constant. In [δ] set $a = \sup\{x : F(x) = 0\}$ and $b = \inf\{x : F(x) = 1\}$. If $a = b$ then X_n is constant a.s.. If $a < b$ then pick $a < m < m' < b$ and use Borel-Cantelli to conclude that almost surely we have $X_n < m$ i.o. and $X_n > m'$ i.o.; thus, $P[X_n(\omega) \not\to] = 1$. If X_n were converging in probability, then a subsequence would converge a.s.; but any subsequence is an i.i.d. sequence and cannot converge a.s.. Alternatively, we note that $P[|X_n - X_m| > \varepsilon] = P[|X_1 - X_2| > \varepsilon]$ for all $n \neq m$ since all pairs (X_n, X_m) with distinct n and m are of equal distribution. So, either $P[|X_n - X_m| > \varepsilon] = 0$ for all n, m or it does not converge to zero.