STAT 582 Homework 4

Due date: In class on Wednesday, March 8, 2006

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13. Let Y_n be independent, Gaussian r.v. with

$$\mathbb{E}[Y_n] = \mu_n \qquad \mathrm{var}(Y_n) = \sigma_n^2.$$

   (a) Assume that $\sum_n \sigma_n^2 < \infty$ and $\sum_n \mu_n$ converge. Show that $\sum_n Y_n$ converges a.s.

   (b) Assume that $\sum_n Y_n$ converges almost surely. Conclude that $\sum_n \sigma_n^2 < \infty$ and $\sum_n \mu_n$ converge.
   Hint: 1st path (long): Use the three-series theorem.
   2nd path (short): Use that $S_n = \sum_{k=1}^{n} Y_k$ converges in distribution. You may use the following result without proving it: *A sequence of $\mathcal{N}(m_n, s_n^2)$ random variables converges in distribution iff $m_n \to m$ and $s_n^2 \to s^2$ in $\mathbb{R}$. If $s^2 = 0$ then the limiting distribution concentrates all probability in the value m, otherwise it is $\mathcal{N}(m, s^2)$.*

From the hint we conclude: A sum of independent Gaussian variables converges a.s. iff it converges in distribution iff the means and the variances are both summable.

We should note that in general we can not conclude from almost sure convergence that moments converge. In the above example μ_n and σ_n are foremost the *parameters* of the random variables. These parameters happen to coincide with moments.

14. Let $f_n(x) = 1 - \cos(2n\pi x)$ for $0 < x < 1$ and zero otherwise.

   (a) Show that $F_n(t) = \int_{-\infty}^{t} f_n(x)dx$ converges vaguely to the uniform distribution on the unit interval, i.e., to the df F with $F(t) = t$ for $0 \le t \le 1$.
   Hint: Show first that $F_n(k/n) = k/n$ for $0 \le k \le n$. This should be relatively easy. Now let $t \in (0, 1)$ be arbitrary, but fixed. We claim that $F_n(t) \to t$. For every n there exists a unique $k \ge 1$ such that $(k-1)/n < t \le k/n$. Use monotonicity of F_n to show that

$$t - 1/n \le F_n(t) \le t + 1/n$$

   and complete the proof.

   (b) Does F_n converge to F in total variation? Show your argument.

   (c) Show that the set of points x where $f_n(x)$ does not converge has positive (non-zero) Lebesgue measure. In other words, it is false that f_n converges Lebesgue-almost everywhere.
   Hint: Assume that f_n converges to g Lebesgue-almost everywhere and derive a contradiction to (a) and/or (b).