19. Let $Y_n$ be independent non-negative random variables bounded by 1, i.e., $0 \leq Y_n \leq 1$ a.s..

(a) Show that if $\sum_n Y_n$ converges a.s. then $\sum \mathbb{E}[Y_n] < \infty$.

(b) Vice versa, show that if $\sum \mathbb{E}[Y_n] < \infty$ then $\sum_n Y_n$ converges a.s.

Let $X_n$ be independent non-negative random variables. Set $Y_n(\omega) = \inf(X_n(\omega), 1)$.

(c) Show that $\sum_n X_n$ converges almost surely if and only if $\sum_n Y_n$ converges almost surely.

Hint for (a) and (b): Three series theorem.
For (c): be careful not to confuse $\inf(X_n(\omega), 1)$ and $X_n I_{|X_n| \leq 1}$.

Note that we established a “one series theorem”:
Let $X_n$ be independent non-negative random variables. Then, $\sum_n X_n$ converges almost surely if and only if $\sum \mathbb{E}[\inf(X_n, 1)] < \infty$.

20. Let $\{X_n\}_n$ be independent random variables. Assume that $\sum_n X_n$ converges in $L^2$.

(a) Show that $\sum_n \mathbb{E}[X_n]$ converges.

(b) Show that $\sum_n X_n$ converges almost surely.

We show now that the reverse of (b) does not hold in all generality.
Let $\{Y_n\}_n$ be independent random variables. Assume that $P[Y_n = n] = 1/n^2$ and $P[Y_n = 0] = 1 - 1/n^2$.

(c) Show that $\sum_n Y_n$ converges almost surely.

(d) Show that $\sum_n Y_n$ does not converge in $L^2$. Hint: use (a).

21. In class we showed convergence in probability implies convergence in distribution. The proof was an explicit verification of the definition of convergence in distribution. Find a new proof using the Portmanteau theorem. More precisely: Assume that $X_n$ converges in probability. Show that one of the equivalent properties of the Portmanteau theorem holds.

Hint: the answer should not take more than one line, quoting yet another (famous) theorem.