STAT 582 Homework 7

Due date: In class on Monday, April 17, 2006

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22. Let $\{X_n\}_n$ be a sequence of Gaussian random variables with means μ_n and variances σ_n^2.

    (a) Assume that $X_n \xrightarrow{\mathrm{D}} X$. Show that necessarily $\mu_n \to \mu$ for some $\mu \in \mathbb{R}$ and $\sigma_n^2 \to \sigma^2$ for some $\sigma^2 \geq 0$.
    Hint: First consider $\rho_n := |\phi_n(1)|$; argue that it must converge. Next show it cannot converge to zero. Conclude that σ_n^2 converges to a real non-negative number.
    Second, turn to μ_n and note that it is the median of the Gaussian r.v. X_n, i.e, $\mu_n = F_n^{\leftarrow}(1/2)$.

    (b) Assume that $\mu_n \to \mu$ for some $\mu \in \mathbb{R}$ and $\sigma_n^2 \to \sigma^2$ for some $\sigma^2 \geq 0$. Show that X_n converges in distribution and describe the limiting distribution in terms of μ and σ^2. Pay attention to the case $\sigma = 0$.

    (c) Give an example of a sequence X_n as above which does not converge in probability.
    Hint: Think what independence would do to the sequence X_n. Also, your example can not have $\sigma^2 = 0$; why?

    (d) Assume now that there is a sequence of independent Gaussian r.v. Y_k and consider $X_n = Y_1 + ... + Y_n$ with the notation from above. Clearly, here σ_n^2 is non-decreasing. Assume that $\mu_n \to \mu$ for some $\mu \in \mathbb{R}$ and $\sigma_n^2 \to \sigma^2$ for some $\sigma^2 \geq 0$. Show that X_n converges almost surely.

    Hint: Continuity and Uniqueness theorem (see Resnick or class notes).

    Note: we showed that a *sum* of *independent* random variables with *Gaussian* distribution converges almost surely iff it converges in distribution iff the sums of means and variances both converge.

    Note: we also showed that a *sequence* of *independent* random variables with *Gaussian* distribution converges in probability iff it converges in distribution to a *constant* iff the means converge and the variances decay to zero. [In the voluntary problem below we study the question whether the sequence could converge even almost surely.]

23. Let $\{B_k\}_n$ be a sequence of independent Bernoulli random variables with $P[B_k = 1] = 1 - P[B_k = 0] = p_k$. Let $M_n = B_1 + \ldots + B_n$ be the associated Binomial variable.

    (a) Show that the characteristic function of M_n is $\phi_n(t) = \prod_{k=1}^{n} (1 + p_k(e^{it} - 1))$.

    (b) Fix $\lambda > 0$. For every given n define M_n as above with $p_k = \lambda/n$ $(k = 1, .., n)$. Now let $n \to \infty$. Show that $M_n \xrightarrow{\mathrm{D}} Y$ for some Poisson r.v. Y. What is the mean of Y? Assume that $p_n = 1/2$ for all n. Let $X_n = 2B_n - 1$.

        i. Show that the characteristic function of X_n is $\cos(t)$.

        ii. Let $S_n = X_1 + \ldots X_n = 2M_n - n$. Show that the characteristic function of $S_n/\sqrt{n}$ is $\phi_n(t) := \cos^n(t/\sqrt{n})$

        iii. Use the approximation $\cos(t) = 1 - t^2/2$ to show that ϕ_n converges to $\exp(-t^2/2)$. Conclude that $S_n/\sqrt{n}$ converges in distribution to a standard normal. Note: this argument is meant to verify the CLT in this special setting. Don't use the CLT for your argument.

    (c) Why are (b) and (c) not in violation of the convergence to types theorem? (After all, (b) and (c) provide different distributional limits of renormalizations of M_n.)

24. (Borel-Cantelli; on popular demand; voluntary]

Let $\{X_n\}_n$ be independent zero mean Gaussian variables with variances $\sigma_n^2 > 0$. We know that $X_n \stackrel{\mathrm{D}}{\to} X$ iff $\sigma_n^2 \to \sigma^2 \geq 0$.

(a) Assume $X_n \stackrel{P}{\to} X$. Show that $\sigma^2 \to 0$ and $X = 0$ a.s.

(b) Assume that $\sigma^2 \to 0$. Show that $X_n \stackrel{P}{\to} 0$.

(c) Assume that $\sigma_n = 1/n$. Show that $X_n \stackrel{\text{a.s.}}{\to} 0$.
Hint: Let Y be standard normal. Note that the distributions of X_n and Y are connected in a simple way. Apply Borel-Cantelli and Markov inequality.

(d) Assume that $\sigma_n \geq \sqrt{2/\log(n)}$. Show that almost surely $X_n(\omega)$ does not converge in $\mathbb{R}$. Thus, $X_n \stackrel{\text{a.s.}}{\not\to} 0$, but $X_n \stackrel{P}{\to} 0$ is possible (i.e., if $\sigma_n \to 0$).
Hint: Let Y be standard normal. Then the following estimate might prove useful for $a > 1$; note that $(a+1)^2 < (2a)^2 = 4a^2$ and so:

$$P[|Y| > a] = \frac{2}{\sqrt{2\pi}} \int_a^\infty e^{-t^2/2} dt \geq \frac{2}{\sqrt{2\pi}} \int_a^{a+1} e^{-t^2/2} dt \geq \frac{2}{\sqrt{2\pi}} e^{-(a+1)^2/2} \geq \frac{2}{\sqrt{2\pi}} e^{-2a^2}$$