STAT 582 Homework 1

Due date: In class on Wednesday, January 31, 2007

Instructor: Dr. Rudolf Riedi

1. (a) Show the following fact.

   Let $\{a_n\}_n$ be a *non-decreasing* sequence of real numbers. If there exists a subsequence $\{a_{n(k)}\}_k$ which converges, then the entire series $\{a_n\}_n$ converges.

   Hint: You may use: (i) a non-decreasing sequence converges if and only if it is bounded. Establish: (ii) $\{a_n\}_n$ is unbounded if and only if every subsequence is unbounded. Combine (i) and (ii) to give the desired proof of the stated fact.

   Note: for an arbitrary sequence the following is true: if the sequence converges then every subsequence converges and vice versa. The point here is that for a monotone sequence one needs only one convergent subsequence to conclude that the entire sequence converges.

   (b) Let X_n be a non-decreasing sequence of random variables. Assume that $X_n \xrightarrow{P} X$. Show that $X_n \overset{\text{a.s.}}{\to} X$.

   Hint: Use (a).

2. Let X_n be a sequence of r.v. Assume that there exists a measurable set N with $P[N] = 0$ with the following property: for every subsequence $\{n(k)\}_k$ there exists a sub-subsequence $\{n(k(i))\}_i$ such that

$$X_{n(k(i))}(\omega) \to X(\omega)$$

   for each $\omega \notin N$. Conclude that $X_n \overset{\text{a.s.}}{\to} X$.

   [Note, that a corollary in class says the following: $X_n \xrightarrow{P} X$ iff for every subsequence $\eta := \{n(k)\}_k$ there exists a measurable set N_η with $P[N_\eta] = 0$ and a sub-subsequence $\{n(k(i))\}_i$ such that $X_{n(k(i))}(\omega) \to X(\omega)$ for each $\omega \notin N_\eta$.

   Now, if the union of all N_η has still zero probability then we can take it as the N in this problem and conclude that we have almost sure convergence. While it does hold sometimes (namely exactly when we have almost sure convergence) this will not hold in general, however. Indeed, there are un-countably many N_η. Also, we know that in general convergence in probability does not imply convergence almost sure.]

3. Give an example of a probability space such that for all $0 < p < 1$ the expression $||X||_p = (\mathbb{E}|X|^p)^{1/p}$ is not a norm, i.e., such that the triangular inequality $||X + Y||_p \leq ||X||_p + ||Y||_p$ fails for some random variables X and Y.

   Hint: Consider a simple coin toss, and set X and Y such that $X + Y = 1$ always.