4. Let $X_n$ be any sequence of random variables. Set $S_n = X_1 + \ldots + X_n$.

(a) Assume that $X_n \xrightarrow{a.s.} 0$. Show that $S_n/n \xrightarrow{a.s.} 0$. Hint: You may use the following fact: Assume that the sequence of real numbers $a_n$ converges to $a$. Then also $(a_1 + \ldots + a_n)/n$ converges to $a$. [By the way, the reverse is not true, as $a_n = (-1)^n$ demonstrates.]

(b) Let $p \geq 1$. Show that if $X_n \xrightarrow{L^p} 0$ then $S_n/n \xrightarrow{L^p} 0$. Hint: triangular inequality.

[Note that $X_n \xrightarrow{P} 0$ does NOT imply $S_n/n \xrightarrow{P} 0$.]

5. Let $X_n$ have normal distribution with mean 0 and variance $\sigma^2_n$.

(a) Assume that $\{X_n\}_n$ is u.i. Show that the variances must be uniformly bounded, i.e., there exists $K$ such that $\sigma_n \leq K$ for all $n$. Hint: study $\mathbb{E}[|X_n|]$.

(b) Assume that the variances are uniformly bounded, i.e., assume there exists $K$ such that $\sigma_n \leq K$ for all $n$. Show that for all $n$

$$
\int_{|X_n|>a} |X_n| dP \leq \frac{2}{\sqrt{2\pi}} K \int_{a/K}^{\infty} y \exp\left(-\frac{y^2}{2}\right) dy
$$

and conclude that $\{X_n\}_n$ is u.i.

(c) Assume that the variances are uniformly bounded. Use the Crystal Ball condition to derive a stronger conclusion than (b), i.e., not only $\{X_n\}_n$ is u.i. but also other families.

In summary, you showed that a sequence of zero mean normal variables is u.i. iff their variances are uniformly bounded.

6. Suppose $\{X_n\}_n$ and $\{Y_n\}_n$ are two u.i. families defined on the same probability space. Is $\{X_n + Y_n\}_n$ u.i.? Show your argument.

Hint: Triangular inequality $|X_n + Y_n| \leq |X_n| + |Y_n|$ a.s.

7. Let $\{X_n\}_n$ be a sequence of i.i.d. random variables with mean zero and variance $\sigma^2$. Let $\{a_n\}_n$ be a sequence of real numbers. Set

$$
S_n = \sum_{i=1}^n a_i X_i.
$$

Prove: $\{S_n\}_n$ converges in $L_2$ $\iff$ $s_n := \sum_{i=1}^n a_i^2$ converges in $\mathbb{R}$.

Hint: you can not use the limiting random variable $\sum_{i=1}^\infty a_i X_i$ to show convergence before you have not established that the limit actually exists. So, use the C... criterium.

8. Let $\{Y_n\}_n$ be a sequence of independent Gaussian random variables with mean zero and variance $\sigma^2_n$. Set $S_n = \sum_{i=1}^n Y_i$. Under what assumptions on the sequence of variances does $S_n$ converge in $L_2$?