STAT 582 Homework 2

Due date: In class on Wednesday, February 7, 2007

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4. Let X_n be any sequence of random variables. Set $S_n = X_1 + \ldots X_n$.

   (a) Assume that $X_n \stackrel{\text{a.s.}}{\to} 0$, Show that $S_n/n \stackrel{\text{a.s.}}{\to} 0$. Hint: You may use the following fact: Assume that the sequence of real numbers a_n converges to a. Then also $(a_1 + \ldots + a_n)/n$ converges to a. [By the way, the reverse is not true, as $a_n = (-1)^n$ demonstrates.]

   (b) Let $p \geq 1$.Show that if $X_n \stackrel{L_p}{\to} 0$ then $S_n/n \stackrel{L_p}{\to} 0$. Hint: triangular inequality.

   [Note that $X_n \stackrel{P}{\to} 0$ does NOT imply $S_n/n \stackrel{P}{\to} 0$.]

5. Let X_n have normal distribution with mean 0 and variance σ_n^2.

   (a) Assume that $\{X_n\}_n$ is u.i. Show that the variances must be uniformly bounded, i.e., there exists K such that $\sigma_n \leq K$ for all n. Hint: study $\mathbb{E}[|X_n|]$.

   (b) Assume that the variances are uniformly bounded, i.e., assume there exists K such that $\sigma_n \leq K$ for all n. Show that for all n

   $$\int_{|X_n|>a} |X_n| dP \leq \frac{2}{\sqrt{2\pi}} K \int_{a/K}^{\infty} y \exp(-\frac{y^2}{2}) dy$$

   and conclude that $\{X_n\}_n$ is u.i.

   (c) Assume that the variances are uniformly bounded. Use the Crystal Ball condition to derive a stronger conclusion than (b), i.e., not only $\{X_n\}$ is u.i. but also other families..

   In summary, you showed that a sequence of zero mean normal variables is u.i. iff their variances are unformly bounded.

6. Suppose $\{X_n\}_n$ and $\{Y_n\}_n$ are two u.i. families defined on the same probability space. Is $\{X_n + Y_n\}_n$ u.i.? Show your argument.

   Hint: Triangular inequality $|X_n + Y_n| \leq |X_n| + |Y_n|$ a.s.

7. Let $\{X_n\}_n$ be a sequence of i.i.d. random variables with mean zero and variance σ^2. Let $\{a_n\}_n$ be a sequence of real numbers. Set

   $$S_n = \sum_{i=1}^{n} a_i X_i.$$

   Prove: $\{S_n\}_n$ converges in $L_2 \iff s_n := \sum_{i=1}^n a_i^2$ converges in $\mathbb{R}$.

   Hint: you can not use the limiting random variable $\sum_{i=1}^{\infty} a_i X_i$ to show convergence before you have not established that the limit actually exists. So, use the C... criterium.

8. Let $\{Y_n\}_n$ be a sequence of independent Gaussian random variables with mean zero and variance σ_n^2. Set $S_n = \sum_{i=1}^n Y_i$. Under what assumptions on the sequence of variances does S_n converge in L_2?