STAT 582 Homework 4

Due date: In class on Wednesday, February 28, 2007

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11. Let Y_n be independent, Gaussian r.v. with

$$\mathbb{E}[Y_n] = \mu_n \qquad \mathrm{var}(Y_n) = \sigma_n^2.$$

   (a) Assume that $\sum_n \sigma_n^2 < \infty$ and $\sum_n \mu_n$ converge. Show that $\sum_n Y_n$ converges a.s.

   (b) Assume that $\sum_n Y_n$ converges almost surely. Conclude that $\sum_n \sigma_n^2 < \infty$ and $\sum_n \mu_n$ converge.
   Hint: 1st path (long): Use the three-series theorem.
   2nd path (short): Use that $S_n = \sum_{k=1}^n Y_k$ converges in distribution. You may use the following result without proving it: *A sequence of $\mathcal{N}(m_n, s_n^2)$ random variables converges in distribution iff $m_n \to m$ and $s_n^2 \to s^2$ in $\mathbb{R}$. If $s^2 = 0$ then the limiting distribution concentrates all probability in the value m, otherwise it is $\mathcal{N}(m, s^2)$.*

   From the hint we conclude: A sum of independent Gaussian variables converges a.s. iff it converges in distribution iff the means and the variances are both summable.

   We should note that in general we can not conclude from almost sure convergence that moments converge. In the above example μ_n and σ_n are foremost the *parameters* of the random variables. These parameters happen to coincide with moments.

12. Let $\{Z_n\}_n$ be a sequence of *independent* exponential random variables with mean one, i.e., $P[Z_n > x] = \exp(-x)$. Let $\{\lambda_n\}_n$ be a sequence of strictly positive numbers. Set $X_n = Z_n/\lambda_n$ and $S_n = X_1 + ... + X_n$.

   (a) Let $c > 0$. Show that

$$\mathbb{E}[X_n \mathbb{I}_{\{X_n < c\}}] = \frac{1}{\lambda_n} - \left(c + \frac{1}{\lambda_n}\right) e^{-c\lambda_n} \tag{1}$$

   (b) Assume that S_n converges a.s. Show that necessarily $\sum_n 1/\lambda_n < \infty$.

   (c) Assume now that $\sum_n 1/\lambda_n < \infty$. Show that S_n converges almost surely.

   Hints:

   (a) Direct computation.

   (b) Proceed along the following steps.
   (i) Use that $P[X_n > x] = \exp(-x\lambda_n)$ to show $\sum_n \exp(-c\lambda_n) < \infty$.
   (ii) Conclude that $\lambda_n \to \infty$.
   (iii) Using this fact conclude that the second additive terms in (1) can be summed over n. Conclude that the first additive terms can be summed, which is the claim.

(c) You may proceed similarly as before in (b) by concluding first that $\lambda_n \to \infty$ and then applying the Two-Series-Theorem.
Alternatively, and maybe even faster, you can apply Kolmogorov's convergence criterium by which you even get convergence in L_2 for free.
Finally, a third (and quickest) possibility is to combine one of the results developed in this set of questions with theorems from class.

13. Here, we establish an extension to the SLLN.

    Assume that $\{X_n\}$ are iid with $\mathbb{E}[X_1^-] < \infty$ and $\mathbb{E}[X_1^+] = \infty$.

    (a) Using Kolmogorov's SLLN show that for any $c > 0$ we have $\liminf_n \bar{X}_n \geq \mathbb{E}[X_1 \mathbb{I}_{\{X_1 < c\}}]$ almost surely.

    (b) Conclude that $\lim_n \bar{X}_n = \infty$ a.s.

    The extended SLLN says then: If $\mathbb{E}[X_1^-]$ and $\mathbb{E}[X_1^+]$ are not both infinite, e.g., if $\mathbb{E}[X_1]$ is well defined, then $\bar{X}_n \to \mathbb{E}[X_1]$.

14. Here, we establish a simplified version of the Three Series Theorem for positive random variables, sometimes called Two-Series Theorem.

    Assume that $\{X_n\}$ are positive. Show that

    if $\sum_n \mathbb{E}[X_n \mathbb{I}_{\{|X_n|<c\}}]$ converges,
    then $\sum_n \mathrm{var}(X_n \mathbb{I}_{\{|X_n|<c\}}) < \infty$.

    In conclusion (Two-Series Theorem): The sum of positive independent r.v. converges almost surely iff the two series (i) and (iii) of the Three Series Theorem converge.

    Hint: Recall that the variance is bounded by the second moment.