# STAT 582 Homework 6

Due date: In class on Wednesday, March 28, 2007

Instructor: Dr. Rudolf Riedi

19. Let $Y_n$ be independent non-negative random variables bounded by 1, i.e., $0 \leq Y_n \leq 1$ a.s..

    (a) Show that if $\sum_n Y_n$ converges a.s. then $\sum \mathbb{E}[Y_n] < \infty$.

    (b) Vice versa, show that if $\sum \mathbb{E}[Y_n] < \infty$ then $\sum_n Y_n$ converges a.s.

    Let $X_n$ be independent non-negative random variables. Set $Y_n(\omega) = \inf(X_n(\omega), 1)$.

    (c) Show that $\sum_n X_n$ converges almost surely if and only if $\sum_n Y_n$ converges almost surely.

    Hint for (a) and (b): Three series theorem.
    For (c): be careful not to confuse $\inf(X_n(\omega), 1)$ and $X_n \mathbb{I}_{|X_n| \leq 1}$.

    Note that we established a "one series theorem":
    Let $X_n$ be independent non-negative random variables. Then, $\sum_n X_n$ converges almost surely if and only if $\sum \mathbb{E}[\inf(X_n, 1)] < \infty$.

20. Let $\{X_n\}_n$ be independent random variables. Assume that $\sum_n X_n$ converges in $L^2$.

    (a) Show that $\sum_n \mathbb{E}[X_n]$ converges.

    (b) Show that $\sum_n X_n$ converges almost surely.

    Hint: Both, (a) and (b) have short solutions.

    We show now that the reverse of (b) does not hold in all generality.
    Let $\{Y_n\}_n$ be independent random variables. Assume that $P[Y_n = n] = 1/n^2$ and $P[Y_n = 0] = 1 - 1/n^2$.

    (c) Show that $\sum_n Y_n$ converges almost surely. Hint: Use Borel-Cantelli or alternatively the Cauchy-criterium.

    (d) Show that $\sum_n Y_n$ does not converge in $L^2$. Hint: use (a).

21. In class we showed convergence in probability implies convergence in distribution. The proof was an explicit verification of the definition of convergence in distribution. Find a new proof using the Portmanteau theorem. More precisely: Assume that $X_n$ converges in probability. Show that one of the equivalent properties of the Portmanteau theorem holds.
    Hint: the answer should not take more than one line, quoting yet another (famous) theorem.