ELEC 533 Homework 5

Due date: In class on Friday, October 11, 2002

Instructor: Dr. Rudolf Riedi

19. The random variable Y represents the number of file requests arriving at a server in a given time. Let X denote the popularity of files in the following sense: if the popularity of all files was equal, say $X = a$ for all files, then Y would be Poisson with mean a. In reality, the popularity X of files varies on a server and has to be taken as a random variable. Still, conditioned on $X = a$ the number of arrivals will be Poisson, i.e., $Y|X = a$ is Poisson with mean a. To summarize the above the following is all you really need to know to solve this problem: The conditional distribution of Y given X is given by

$$P[Y = k|X = a] = \frac{e^{-a}a^k}{k!}$$

For simplicity[1] we assume here that X is a uniform random variable as in

$$f_X(x) = \begin{cases} 1/5 & 0 \leq x \leq 5 \\ 0 & \text{else} \end{cases}$$

Obviously, Y is not Poisson. Find the unconditional distribution $P[Y = k]$ using the Law of Total Probability

$$P[Y = k] = \int_{-\infty}^{\infty} P[Y = k|X = x] f_X(x) dx$$

- *Hint :* Use integration by parts to establish a recursive formula which allows to compute $P[Y = k]$ from $P[Y = k - 1]$. Compute $P[Y = 0]$ explicitly. Putting the two together gives an explicit expression for $P[Y = k]$.

20. Consider the experiment of tossing three coins independently. The outcomes are then $\omega = (\omega_1, \omega_2, \omega_3) \in \Omega = \{0, 1\}^3$, where ω_n is the outcome of the n-th toss, $\omega_n = 1$ for heads, $\omega_n = 0$ for tails. Let $X_1(\omega) = \omega_1$, $X_2(\omega) = \omega_2$, $X_3(\omega) = \omega_3$, $Y = X_1 + X_2$ and $Z = X_1 + X_2 + X_3$.

    (a) Compute $\mathbb{E}[Z|Y]$.

    (b) Using the conditional probability law $P[Z = n|X_1 = x_1 \text{ and } X_2 = x_2]$ we define the conditional expectation in the usual (intuitive) way as

    $$\mathbb{E}[Z|X_1 = x_1, X_2 = x_2] := \sum_n n \cdot P[Z = n|X_1 = x_1, X_2 = x_2].$$

    Compute this expression as a function $h(x_1, x_2)$. We set $\mathbb{E}[Z|X_1, X_2] = h(X_1, X_2)$.

    (c) Show that $\mathbb{E}[Z|Y] = \mathbb{E}[Z|X_1, X_2]$. This means that Y gives the same information towards predicting Z as X_1 and X_2.

    (d) Compute $\mathbb{E}[Z|X_1]$.

    (e) Consider the random variable $U = \mathbb{E}[Z|X_1, X_2]$. Compute $\mathbb{E}[U|X_1]$. Compare with 20d.

    (f) Compute the variance of the random variables $Z - \mathbb{E}[Z|X_1]$ and $Z - \mathbb{E}[Z|X_1, X_2]$ and compare them.
    This illustrates that the variance of error grows as we take a guess at Z with less knowledge.

[1] A more realistic assumption would be a Zipf law for X, i.e., a power law decay for the density of X.

21. Let X and Y be standard jointly Gaussian r.v.'s ($\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$) with joint density

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left(-\frac{x^2 - 2\rho xy + y^2}{2(1-\rho^2)}\right),$$

where ρ is a constant.

(a) Show by direct computation that $\mathrm{cov}(X, Y)/\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)} = \rho$.

(b) Compute the conditional density $f_{Y|X=a}$ using $f_{Y|X=a}(y) = f_{XY}(a, y)/f_X(a)$.

(c) Compute the conditional expectation $\mathbb{E}[Y|X]$ using above density.

(d) Compute the density f_Z of the sum $Z = X + Y$ using the general formula for the density of the sum of (dependent) random variables: $f_Z(z) = \int f_{XY}(x, z - x)dx$. Conclude that Z is as well Gaussian.

(e) Let (U, V) be joint continuous random variables with joint density f_{UV}. Let $W = U + V$. Derive the following formula for the joint density of U and W:

$$f_{UW}(a, c) = f_{UV}(a, c - a). \tag{1}$$

Hint: Express first the joint CDF $F_{UW}(a, b) = P[U \leq a, W \leq b]$ as an appropriate integral using f_{UV}. Then, take derivatives.

(f) Using formula (1) show that U and W are jointly Gaussian, provided U and V are jointly Gaussian. For simplicity assume that U and V have mean zero and variance 1.