STAT 650 Homework 1

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Due date: Thursday, Feb. 9, 2006
Bring to class or hand in to Dr. Riedi, Duncan Hall 2082

1. Let U be a discrete random variable with values $\{u_i\}_{i\in I}$. Let $A_i := \{\omega : U(\omega) = u_i\}$. Note that the sets A_i form a partition of Ω, and that $\sigma(U) = \sigma(A_i, i \in I)$. for ease of notation we write $\mathbb{E}[Y|U] = \mathbb{E}[Y|\sigma(U)]$.

   Let Y be in L^1. Recall that $\mathbb{E}[Y|U]$ is uniquely determined (in the sense of a.s. equality) by two properties, i.e., a r.v. Z is a.s. equal to $\mathbb{E}[Y|U]$ if and only if

   ((i)) Z is $\sigma(U)$-measurable

   ((ii)) $\mathbb{E}[\mathbb{1}_A \cdot Z] = \mathbb{E}[\mathbb{1}_A \cdot Y]$ for all $A \in \sigma(U)$

   Verify the following facts using *only* the definition of a conditional expectations, i.e., the properties ((i)) and ((ii)) above. Thereby note that you need to verify ((ii)) only for the generating sets $A = A_i$.

   (a) Show that there exist real numbers a_i such that

   $$\mathbb{E}[Y|U] = \sum_{i\in I} a_i \mathbb{1}_{A_i}(\omega) \tag{1}$$

   where the a_i must satisfy the condition $\mathbb{E}[a_i \mathbb{1}_{A_i}] = \mathbb{E}[Y \mathbb{1}_{A_i}]$ and can otherwise be chosen arbitrarily (note the case $P[A_i] = 0$). More precisely, set $Z = \sum_{i\in I} a_i \mathbb{1}_{A_i}(\omega)$ and verify ((i)) and ((ii)).

   (b) Assume that Y is independent of U. Show that

   $$\mathbb{E}[Y|U] = \mathbb{E}[Y].$$

   More precisely, set $Z = \mathbb{E}[Y]$ and verify ((i)) and ((ii)).

   (c) Let $h(x, y)$ be measurable and assume that $h(U, Y)$ lies in L^1. Due to (a) we know that the following form must hold:

   $$\mathbb{E}[h(U, Y)|U] = \sum_{i\in I} \alpha_i \mathbb{1}_{A_i}(\omega).$$

   Compute α_i.

   (d) Let h be as above. Show the following fact. If Y and U are independent, then we have $\alpha_i P(A_i) = \mathbb{E}[h(u_i, Y)]P(A_i)$ for all i. In other words,

   $$\mathbb{E}[h(U, Y)|U] = f(U) \qquad \text{where } f(x) = \mathbb{E}[h(x, Y)].$$

   More precisely, set $Z = f(U)$ and verify ((ii)). You may assume that ((i)) holds.

   (e) Let h be as above. Show the following fact. If g is measurable such that $g(U)Y$ is in L^1 then

   $$\mathbb{E}[g(U)Y|U] = g(U)\mathbb{E}[Y|U]$$

   More precisely, set $Z = g(U)\mathbb{E}[Y|U]$ and verify ((i)) and ((ii)).

2. Show that $X_t = W_t^2 - t$ is a martingale w.r.t. $\mathcal{W}_t := \sigma(W_s : s \leq t)$.
   Hint: Write $W_t = W_s + (W_t - W_s)$ for $s < t$.

3. Notation:

   Generic partition π of a compact interval $[S,T]$: $\pi = \{t_j\}_{j=0}^N$ where $S = t_0 \leq t_1 \leq ... \leq t_N = T$. The partition intervals $I_j = [t_j, t_{j+1})$ have length $\Delta_j = (t_{j+1} - t_j)$ and indicator functions:

$$\mathbb{1}_j(t) = \begin{cases} 1 & \text{if } t_j \leq t < t_{j+1} \\ 0 & \text{else.} \end{cases}$$

   Granularity: $||\pi|| := \max\{\Delta_j : 0 \leq j < N\}$.
   Corresponding Brownian increments: $\Delta W_j = W(t_{j+1}) - W(t_j)$.

   (a) From a result in class it follows that

$$\{Z_\pi(t,\omega)\}_t := \left\{ \sum_{j=0}^{N-1} W(t_j,\omega)\mathbb{1}_j(t) \right\}_t \xrightarrow{\mathcal{V},L_2} \{W(t,\omega)\}_t \qquad \text{as } ||\pi|| \to 0. \tag{2}$$

   Verify this explicitly, i.e., show that $\mathbb{E}\int_S^T |Z_\pi(t,\omega) - W(t,\omega)|^2 dt \to 0$ as $||\pi|| \to 0$.
   Hint: write $\int_S^T ... = \sum_j \int_{I_j} ...$ and note that $\mathbb{1}_k(t)$ vanishes identically over I_j unless $j = k$.

   (b) In class we saw that

$$\int_0^t W(s)dW(s) := \lim W(t_j)\Delta W_j = \frac{1}{2}(W^2(t) - t) \tag{3}$$

   Generalize this result as follows. Let $\theta \in [0,1]$. Then,

$$\int_0^t W(s) * dW(s) := \lim W(t_j + \theta \cdot \Delta_j)\Delta W_j = \frac{1}{2}W^2(t) + (\theta - 1/2) \cdot t \tag{4}$$

   This means that the Stratonovich version of the integral (3) which is obtained by setting $\theta = 1/2$, becomes:

$$\int_0^t W(s) \circ dW(s) := \lim W(\frac{t_j + t_{j+1}}{2})\Delta W_j = \frac{1}{2}W^2(t) \tag{5}$$

   Also, it means that $\mathbb{E}[\int_0^t W(s) * dW(s)] = \theta \cdot t$, which we obtained in class for $\theta = 0$ and $\theta = 1$.

   Hint: Exploiting the known fact (3) from class it is enough to show that $\int_0^t W(s) * dW(s) - \int_0^t W(s)dW(s) = \theta \cdot t$. To this end, show that

$$\sum_j \Delta_\theta W_j \Delta W_j - \theta \cdot t \quad \overset{L_2}{\to} \quad 0 \qquad \text{as } ||\pi|| \to 0.$$

   Here we set $\Delta_\theta W_j = W(t_j + \theta \cdot \Delta_j) - W(t_j)$. It helps to introduce $\Delta_\theta^* W_j = W(t_{j+1}) - W(t_j + \theta \cdot \Delta_j)$, so that $\Delta_\theta W_j + \Delta_\theta^* W_j = \Delta W_j$ and to exploit independence of increments over disjoint intervals. Also, write $t = \sum_j \Delta_j$ is in class. Recall that the 4th moment of a zero-mean Gaussian variable is three times the square of the variance.