STAT 655
Extra Homework Exercise
due 21 Mar 2006 with 7.1, 7.2, 7.8 from book

Let

$$C_B(I\!R^n) \; = \; \{f : f \text{ is bounded and continuous from } I\!R^n \longrightarrow I\!R\}.$$

For $f \in C_B(I\!R^n)$ define the norm

$$\|f\| = \sup_{x \in R^n} |f(x)|.$$

For $t \geq 0$ and f a bounded Borel function, define the operators

$$T_t f(x) \; = \; E^x[f(X_t)],$$

where X_t is an Ito diffusion and x is a starting value. The idea here is that $T_t f$ is a function of the starting value x. We set T_0 to be the identity: $T_0 f = f$.

Prove the following

(i) For $f \in C_B(I\!R^n)$, $T_t f \in C_B(I\!R^n)$.

(ii) The T_t is linear.

(iii) $\|T_t\| \stackrel{def}{=} \sup\{\|T_t f\| : f \in C_B(I\!R) \ \& \ \|f\| = 1\} \; = \; 1.$

(iv) $T_t T_s = T_{t+s}$.

Hints: For part (1), look at the Proof of Thm. 5.2.1 in the book. From that, you can get the inequality

$$E\left[\left|X_t^x - X_t^{x'}\right|\right] \; \leq \; K_t |x - x'|,$$

where K_t depends on t but not x nor x'. From this, and simple facts about probability distributions, you can argue that for any $\epsilon > 0$, there is a compact set $K \subset I\!\!R^n$

$$P\left[X_t^x \in K \,\&\, X_t^{x'} \in K \,\&\, \left|X_t^x - X_t^{x'}\right| \le \epsilon\right] \ge 1 - \epsilon.$$

Recall that a continuous function is uniformly continuous on a compact set. Parts (ii) and (iii) are easy. Part (iv) is a restatement of the Markov property, in some sense. Part (iv) is known as the "semigroup" property. There is a somewhat extensive theory of operator semigroups which was connected to probability by Feller, Doob, Yoshida, Blumenthal, Kurtz, and others. Having this property means that any of the T_t operators is very special: T_t has a "square-root" $T_{t/2}$, which makes it a "positive" operator, but its square root also has a square root, etc.

Indeed, the semigroup is characterized by it's (infinitesimal) generator A, which in some sense is dT_t/dt at $t = 0$. Unfortunately, A is not generally a bounded operator (as in item (iii) above), but it still works to characterize the semigroup and provides (more or less) practical ways of computing various expectations and transition probability density functions using partial differential equations, as we shall see. From another view, it provides methods for solving PDEs using simulation.

One other issue from class March 9:

Theorem Suppose X_t is a *cadlag* process adapted to a complete filtration $\mathcal{F}_t$. If τ is any finite stopping time, then $X_\tau \in \mathcal{F}_\tau$.

Discussion & Proof. Cadlag means X_t is right continuous with left limits. For a lot of reasons, this is the right kind of process to consider. For example, there is a lot of interest in "jump diffusions" in finance nowadays. These processes are Markovian and wander about generally in a continuous way, but sometimes jump. To have a Markov property that is meaningful, we need to know where they jump to at exactly the jump time. So, we need that $X_{t+h} \to X_t$ as $h \downarrow 0$, i.e. right continuity. The left limit requirement is less obvious but will hold in general for most practical models.

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Writing $X_\tau \in \mathcal{F}_\tau$ means X_τ is $\mathcal{F}_\tau$-measurable.

Proof: We want to show that for all $t \geq 0$ and Borel sets B,

$$[X_\tau \in B] \cap [\tau \leq t] \in \mathcal{F}_t.$$

One can show that the collection of sets B satisfying this relation is a σ-field, so it suffices to consider just B open. Now for any $s \leq t$, we have

$$
\begin{aligned}
[X_\tau \in B] \cap [\tau = s] \quad &= \quad [X_s \in B] \cap [\tau = s] \\
&\in \quad \mathcal{F}_s \\
&\subset \quad \mathcal{F}_t.
\end{aligned}
\tag{1}
$$

(Note that the relation $[\tau = s] \in \mathcal{F}_s$ was proven in lecture.) However, we can't make use of this fact very well since

$$[X_\tau \in B] \cap [\tau \leq t] \quad = \quad \bigcup_{s \leq t} ([X_s \in B] \cap [\tau = s])$$

involves an uncountable union. We will have to use the result of (1) when $s = t$ since we have no information to the right of t but for $s < t$, using the fact that B is open and X_t is right continuous, we have that

$$
\begin{aligned}
&([X_\tau \in B] \& [\tau \leq t]) \\
&\quad \Longleftrightarrow \quad (\exists q \in Q \cap [0,t) \text{ s.t. } \tau < q \,\&\, \forall q' \in Q \cap [0,q),\, \tau \leq q' \Longrightarrow X_{q'} \in B.)
\end{aligned}
$$

Here, Q denotes the rational numbers. Recall that $A \Longrightarrow B$ is logically equivalent to $\neg A \vee B$, where $\neg$ means negation and $\vee$ means "*or*." Translating this into sets, we have

$$
\begin{aligned}
[X_\tau \in B] \cap [\tau \leq t] \quad &= \quad ([X_t \in B] \cap [\tau = t]) \\
&\quad \cup \bigcup_{q \in Q \cap [0,t)} \bigcap_{q' \in Q \cap [0,q)} \{[q' < \tau < q] \cup [X_{q'} \in B]\})
\end{aligned}
$$

We see that all the events in the expression are in $\mathcal{F}_t$ and that the set operations are countable so the right hand side is in $\mathcal{F}_t$.

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