1. Let \( \{Z(t)\}_t \) be a stochastic process. Recall that the process \( Z \) has stationary increments iff
\[
\{Z(t)\}_t \quad \text{and} \quad \{Z(t+h) - Z(t)\}_h \text{ are identically distributed (i.d.)} \quad (1)
\]
(a) Show that stationary increments imply that for all sample points \( t_1 < s_1 < t_2 < s_3 < \ldots < t_n < s_n \) we have
\[
(Z(t+s_1)-Z(t+t_1), \ldots, Z(t+s_n)-Z(t+t_n)) \overset{d}{=} (Z(s_1)-Z(t_1), \ldots, Z(s_n)-Z(t_n)) \quad (2)
\]
Hint: we could even include the increments \( Z(t+t_k)-Z(t+s_{k-1}) \) and \( Z(t_k)-Z(s_{k-1}) \).
(b) Show that stationary increments imply that the increment processes \( Y^{(r)}(\tau) \) are stationary for all \( \tau \) where \( \{Y^{(r)}(t)\}_t := \{Z(t+\tau)-Z(t)\}_t \). Hint: use (2) when comparing the f.d.d. of \( Y^{(r)}(\tau) \) and \( Y^{(r)}(\cdot + h) \). For easy of notation write \( Y \) instead of \( Y^{(r)} \) when the choice of \( \tau \) is clear or when \( \tau \) is arbitrary.
(c) Assume that the increment processes \( Y^{(r)}(\tau) \) are stationary. Show that (2) holds for any sample points \( t_1 < s_1, \ldots, t_n < s_n \) such that one can find \( \tau \) with \( (s_k - t_k)/\tau \in \mathbb{Z} \).
(d) Conclude for a process indexed by rational times: \( \{Z(t)\}_{t \in \mathbb{R}} \) has stationary increments iff its increment processes \( Y^{(r)}(\tau) \) are stationary.
(e) Conclude: Assume that \( \{Z(t)\}_{t \in \mathbb{R}} \) has almost surely continuous paths. Then, \( \{Z(t)\}_{t \in \mathbb{R}} \) has stationary increments iff its increment processes \( Y^{(r)}(\tau) \) are stationary.

2. Let \( X_t(\omega) = X(t, \omega) = X(t) \) for all \( t \) and \( \omega \). Show that \( X_t \) is \( H \)-sssi with \( H = 1 \).
Assume that \( \{Z(t)\}_{t \in \mathbb{R}} \) is \( H \)-sssi, \( 0 < H \geq 1 \), with \( Z(1) = 1 \). Show that for each \( t \) the random variable \( Z(t) \) is almost surely constant. What is the value of that constant as a function of \( t \)?

3. Linear Sequence
Recall the convolution operator which maps two series \( c := \{c_j\}_j \) and \( d := \{d_j\}_j \) to a new series \( c * d \) defined by
\[
(c * d)_k := \sum_{j \in \mathbb{Z}} c_{k-j}d_j = \sum_{j \in \mathbb{Z}} c_j d_{k-j}
\]
(a) Auto-covariance
Let \( \{X_k\}_k \) be a linear sequence of the form
\[
X_k = c * \varepsilon = \varepsilon * c \quad (3)
\]
where \( c = \{c_j\}_j \) is in \( l_2 \), i.e., \( \sum_j |c_j|^2 < \infty \), and where \( \varepsilon = \{\varepsilon_j\}_j \) is a sequence of i.i.d. random variables with finite variance \( \sigma^2 \) and zero mean.
Show that the auto-covariance of \( X \) is
\[
\gamma_X(k) = \mathbb{E}[X_kX_{i+k}] = \sigma^2 \sum_{j \in \mathbb{Z}} c_j c_{k+j}
\]
in other words, \( \{\gamma(k)\}_k = \sigma^2 \cdot c \cdot c^{\perp} \) where \( c^{\perp} = c_{-j} \). Note, this also implies that \( X \) is second-order stationary and that \( \gamma_X(k) = \gamma_X(-k) \).
(b) [Exponential decay of correlations for AR series]

Assume now that \( \{X_k \}_k \) is FARIMA(1,0,0), in other words an AR(1) process, and satisfies an auto-regressive invariance of the form

\[
X_k = (1 - \phi B)^{-1} \varepsilon_k = \sum_{j \geq 0} \phi^j \varepsilon_{k-j}
\]

Again in other words, \( \{X_k \}_k \) is a linear sequence of the form \( \phi \ast \varepsilon \) with \( \phi = \{\phi^j\}_j \).

Determine the range of real numbers \( \phi \) for which this makes sense. For such \( \phi \), show that the auto-covariance is \( \gamma_X(k) = \phi^k \sigma^2 / (1 - \phi^2) \) and thus decays exponentially fast.

(c) [FARIMA(0,d,0) series]

Assume now that \( \{X_k \}_k \) is FARIMA(0,d,0) for some \(-1/2 < d < 1/2 \) with \( d \neq 0 \), and satisfies an auto-regressive invariance of the form

\[
X_k = (1 - B)^{-d} \varepsilon_k = \sum_{j \geq 0} b_j B^j \varepsilon_k = \sum_{j \geq 0} b_j \varepsilon_{k-j}
\]

where

\[
b_j = \frac{\Gamma(d+j)}{\Gamma(j+1)\Gamma(d)} \frac{(d+j-1)(d+j-2)...(d+1)d}{j!} \quad (j \geq 0)
\]

Use Stirling’s formula \( \Gamma(x+1) \sim \sqrt{2\pi e^{-x}(x+1)^{x+1/2}} \) and the well known \( (1 + x/j)^j \rightarrow e^x \) to show that \( b_j \cdot j^{1-d} \rightarrow \eta \) as \( j \rightarrow \infty \) for some constant \( \eta \).

Conclude that the FARIMA(0,d,0) is well-defined in \( L^2 \) and almost surely for \( d < 1/2 \).

(d) [Power-law decay of correlations for FARIMA(0,d,0) series]

Assume now that \( \{X_k \}_k \) is FARIMA(0,d,0) as in (3c). Use (3a) to conclude that the auto-covariance decays as

\[
\gamma(|k|) \sim |1/k|^{1-2d}
\]

Conclude that FARIMA(0,d,0) exhibits LRD for \( 0 < d < 1/2 \) with the same auto-covariance decay as the increment process of an H-ssi process with \( H = d + 1/2 \).

Hint: You may use a fact similar to Prop 4.1 of the lecture notes saying that if a sequence \( c \) is positive and ultimately monotone, then we have: \( \hat{c}(\nu) \sim |\nu|^{-d} \) if and only if \( c_j + c_j \sim (1/j)^{1-d} \) \( (|j| \rightarrow \infty) \). The point is that here we may know the precise decay of \( c \) at one side only. Here, \( \hat{c} \) denotes the spectral density or Fourier transform of \( c \). Note the simple relation between \( \hat{c} \), \( \hat{c}^- \) and \( \hat{c} + \hat{c}^- \).

(e) [FARIMA(1,d,0) series]

Assume finally that \( \{X_k \}_k \) is FARIMA(1,d,0) for some \(-1/2 < d < 1/2 \) with \( d \neq 0 \), and satisfies an auto-regressive invariance of the form

\[
X_k = (1 - \phi B)^{-1}(1 - B)^{-d} \varepsilon_k
\]

Show that this is a linear sequence of the form \( X = d \ast \varepsilon \) and express \( d \) in terms of \( \phi \) and \( b_k \) from (3c). Does the order in which the two “filters” are applied to the noise play a role?

Hint: Express \( d = \{d_j\}_j \) first in terms of convolutions, then compute it.