STAT 551 Homework 1

Due date: In class on Monday, November 6, 2006

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1. Let $\{Z(t)\}_t$ be a stochastic process. Recall that the process Z has stationary increments iff

$$\{Z(t+h) - Z(t)\}_h \overset{\text{fdd}}{=} \{Z(h) - Z(0)\}_h \tag{1}$$

   (a) Show that stationary increments imply that for all sample points $t_1 < s_1 < t_2 < s_3 < ... < t_n < s_n$ we have

$$(Z(t+s_1) - Z(t+t_1), ..., Z(t+s_n) - Z(t+t_n)) \overset{\text{d}}{=} (Z(s_1) - Z(t_1), ..., Z(s_n) - Z(t_n)) \tag{2}$$

   Hint: we could even include the increments $Z(t+t_k) - Z(t+s_{k-1})$ and $Z(t_k) - Z(s_{k-1})$.

   (b) Show that stationary increments imply that the increment processes $Y^{(\tau)}$ are stationary for all τ where $\{Y^{(\tau)}(t)\}_t := \{Z(t+\tau) - Z(t)\}_t$.
   Hint: use (2) when comparing the f.d.d. of $Y^{(\tau)}$ and $Y^{(\tau)}(\cdot + h)$. For easy of notation write Y instead of $Y^{(\tau)}$ when the choice of τ is clear or when τ is arbitrary.

   (c) Assume that the increment processes $Y^{(\tau)}$ are stationary. Show that (2) holds for any sample points $t_1 < s_1, ..., t_n < s_n$ such that one can find τ with $(s_k - t_k)/\tau \in \mathbb{Z}$.

   (d) Conclude for a process indexed by rational times: $\{Z(t)\}_{t \in \mathbb{Q}}$ has stationary increments iff its increment processes $Y^{(\tau)}$ are stationary.

   (e) Conclude: Assume that $\{Z(t)\}_{t \in \mathbb{R}}$ has almost surely continuous paths. Then, $\{Z(t)\}_{t \in \mathbb{R}}$ has stationary increments iff its increment processes $Y^{(\tau)}$ are stationary.

2. Let $X_t(\omega) = X(t, \omega) = X(t)) = t$ for all t and ω. Show that X_t is H-sssi with $H = 1$.

   Assume that $\{Z(t)\}_{t \in \mathbb{R}}$ is H-sssi, $0 < H \geq 1$, with $Z(1) = 1$. Show that for each t the random variable $Z(t)$ is almost surely constant. What is the value of that constant as a function of t?

3. [Linear Sequence]

   Recall the convolution operator which maps two series $c := \{c_j\}_j$ and $d := \{d_j\}_j$ to a new series $c * d$ defined by

$$(c * d)_k := \sum_{j \in \mathbb{Z}} c_{k-j} d_j = \sum_{j \in \mathbb{Z}} c_j d_{k-j}$$

   (a) [Auto-covariance]
   Let $\{X_k\}_k$ be a linear sequence of the form

$$X_k = c * \varepsilon = \varepsilon * c \tag{3}$$

   where $c = \{c_j\}_j$ is in l_2, i.e., $\sum_j |c_j|^2 < \infty$, and where $\varepsilon = \{\varepsilon_j\}_j$ is a sequence of i.i.d. random variables with finite variance σ^2 and zero mean.
   Show that the auto-covariance of X is

$$\gamma_X(k) = \mathbb{E}[X_i X_{i+k}] = \sigma^2 \sum_{j \in \mathbb{Z}} c_j c_{k+j}$$

   in other words, $\{\gamma(k)\}_k = \sigma^2 \cdot c * c^-$ where $c_j^- = c_{-j}$. Note, this also implies that X is second-order stationary and that $\gamma_X(k) = \gamma_X(-k)$.

(b) [Exponential decay of correlations for AR series]
Assume now that $\{X_k\}_k$ is FARIMA(1,0,0), in other words an AR(1) process, and satisfies an auto-regressive invariance of the form

$$X_k = (1-\phi B)^{-1}\varepsilon_k = \sum_{j\geq 0}(\phi B)^j \varepsilon_k = \sum_{j\geq 0}\phi^j \varepsilon_{k-j}$$

Again in other words, $\{X_k\}_k$ is a linear sequence of the form $\phi * \varepsilon$ with $\phi = \{\phi^j\}_j$. Determine the range of real numbers ϕ for which this makes sense. For such ϕ, show that the auto-covariance is $\gamma_X(k) = \phi^k\sigma^2/(1-\phi^2)$ and thus decays exponentially fast.

(c) [FARIMA(0,d,0) series]
Assume now that $\{X_k\}_k$ is FARIMA(0,d,0) for some $-1/2 < d < 1/2$ with $d \neq 0$, and satisfies an auto-regressive invariance of the form

$$X_k = (1-B)^{-d}\varepsilon_k = \sum_{j\geq 0} b_j B^j \varepsilon_k = \sum_{j\geq 0} b_j \varepsilon_{k-j}$$

where

$$b_j = \frac{\Gamma(d+j)}{\Gamma(j+1)\Gamma(d)} = \frac{(d+j-1)(d+j-2)...(d+1)d}{j!} \qquad (j \geq 0)$$

Use Stirling's formula $\Gamma(x+1) \sim \sqrt{2\pi}e^{-x}x^{x+1/2}$ and the well known $(1+x/j)^j \to e^x$ to show that $b_j \cdot j^{1-d} \to \eta$ as $j \to \infty$ for some constant η.
Conclude that the FARIMA(0,d,0) is well-defined in L_2 and almost surely for $d < 1/2$.

(d) [Power-law decay of correlations for FARIMA(0,d,0) series]
Assume $\{X_k\}_k$ is FARIMA(0,d,0) as in (3c). Use (3a) to conclude that the auto-covariance decays as

$$\gamma(|k|) \sim |1/k|^{1-2d}$$

Conclude that FARIMA(0,d,0) exhibits LRD for $0 < d < 1/2$ with the same auto-covariance decay as the increment process of an H-sssi process with $H = d + 1/2$.
Hint: You may use a fact similar to Prop 4.1 of the lecture notes saying that if a sequence c is positive and ultimately monotone, then we have: $\hat{c}(\nu) \sim |\nu|^{-d}$ $(\nu \to 0)$ if and only if $c_j + c_j^- \sim (1/j)^{1-d}$ $(|j| \to \infty)$. The point is that here we may know the precise decay of c at one side only. Here, $\hat{c}$ denotes the spectral density or Fourier transform of c. Note the simple relation between $\widehat{c}$, $\widehat{c^-}$ and $\widehat{c+c^-}$.

(e) [FARIMA(1,d,0) series]
Assume finally that $\{X_k\}_k$ is FARIMA(1,d,0) for some $-1/2 < d < 1/2$ with $d \neq 0$, and satisfies an auto-regressive invariance of the form

$$X_k = (1-\phi B)^{-1}(1-B)^{-d}\varepsilon_k$$

Show that this is a linear sequence of the form $X = d * \varepsilon$ and express d_j in terms of ϕ and b_k from (3c). Does the order in which the two "filters" are applied to the noise play a role?
Hint: Express $d = \{d_j\}_j$ first in terms of convolutions, then compute it.