Student's t-test for Scale Mixture Errors

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Generalized t-tests are constructed under weaker than normal conditions. First we assume the errors are scale mixtures of normal random variables (which includes many heavy tailed distributions) and compute the critical values of the suggested s-test. This s-test is optimal in the sense that if the level is α, the s-test provides the minimal critical values in this class. The s-test turned out to be a t-type test but its degree of freedom depends on α.

Denote the counterpart of Student's $t_n(x)$ by $s_n(x)$ and define the scale mixture counterpart of the standard normal cdf as follows. $\Phi^*(x) := \lim_{n\to\infty} s_n(x)$. We prove that there is a sequence $1 = r(1) < r(2) < \ldots < \sqrt{3}$ such that

$$\Phi^*(x) = \begin{cases} .5 & x \in [0,1) \\ P(t_{k-1} \le x\sqrt{(k-1)/(k-x^2)}) & x \in [r(k-1), r(k)), \quad k = 2,3,\ldots \\ \Phi(x) & x \ge \sqrt{3} \quad (\Phi^*(\sqrt{3}) = \Phi(\sqrt{3}) = 0.958). \end{cases}$$

E.g. if $x \in [r(2), r(3)) = [1.3136, 1.4282)$, then $\Phi^*(x) = P(t_2 \le x\sqrt{2/(3-x^2)}) = 1/2 + x/(2\sqrt{3})$ (the .9 quantile is in this interval). Our theorem shows how robust the z-test is for $x \ge \sqrt{3}$. On the other hand, if $1-\alpha < \Phi^*(\sqrt{3}) = \Phi(\sqrt{3}) = 0.958\ldots$, then $\Phi^*(x)$ should be applied for computing critical values of the counterpart of the z-test.

The s-test is applicable for many heavy tailed errors e.g. in bioinformatics and in the stochastic volatility geometric Brownian motion model for stock prices. The assumption on the error distribution can be made even weaker supposing only that the error distribution is symmetric and unimodal. The results also show that the usual confidence intervals of typical "media statistics" are not quite reliable.