STAT 331 Fall '05: MIDTERM SOLUTIONS

1. Short Questions

a.

$$\begin{aligned} E[V] &= \int_0^4 \frac{4}{3}\pi r^3 f(r)\,dr \\ &= \int_0^4 \frac{\pi}{6} r^4\,dr \\ &= \frac{\pi}{6}\left[\frac{r^5}{5}\right]_0^4 \\ &= \frac{512\pi}{15} \approx 107. \end{aligned}$$

Some people computed $E[V] = \frac{4}{3}\pi E[R^3]$, but keep in mind it is generally not true that $E[g(X)] = g(E[X])$ unless g is linear.

b. Let X be the total number of errors in the sequence of n information bits, and let

$$X_i = \begin{cases} 1 & \text{if the } i\text{th information bit is decoded incorrectly} \\ 0 & \text{otherwise} \end{cases}$$

so that $X = \sum_{i=1}^n X_i$. Denote $\tilde{p} = P(X_i = 1)$, so that $X_i \sim \text{Ber}(\tilde{p})$ and $X \sim \text{bin}(n, \tilde{p})$. Then

$$\begin{aligned} P(\text{an error}) &= 1 - P(\text{no error}) \\ &= 1 - P(X = 0) \\ &= 1 - (1 - \tilde{p})^n, \end{aligned}$$

so it remains to find $\tilde{p}$. Note that $X_i = 1$ if and only if 3 or more of the five repeated bits are corrupted. Thus

$$\begin{aligned} \tilde{p} &= P(3, 4, \text{ or } 5 \text{ flipped bits out of } 5) \\ &= \binom{5}{3} p^3 (1-p)^2 + \binom{5}{4} p^4 (1-p) + \binom{5}{5} p^5 \\ &= 0.0579 \end{aligned}$$

where I have plugged in $p = 0.2$. In conclusion, $P(\text{an error}) = 1 - (0.94)^n$.

c. Let X be the point where the stick is broken. Then $X \sim \text{unif}[0, 1]$. The probability of interest is

$$\begin{aligned} P(X \in [0, 1/3] \cup [2/3, 1]) &= F(1/3) - F(0) + F(1) - F(2/3) \\ &= 1/3 + 1/3 \\ &= 2/3 \end{aligned}$$

where $F(x) = x$ is the CDF of X. **Note:** Many people looked at the random variable $Y = \max(X, 1 - X)$, the length of the longest piece, and computed $P(Y \geq 2/3)$. This is fine in principle. However, you cannot simply say that $Y \sim \text{unif}[1/2, 1]$ without any justifcation. While true, this must shown by, for example, finding the CDF of Y and identifying it as that of a uniform.

d.

$$\begin{aligned}
P(1 \le Y \le 3) &= P(\ln 1 \le X \le \ln 3) \\
&= F_X(\ln 3) - F_X(\ln 1) \\
&= \Phi\left(\frac{\ln 3 - 1}{\sqrt{2}}\right) - \Phi\left(\frac{\ln 1 - 1}{\sqrt{2}}\right) \\
&= \Phi(0.0679) - \Phi(-0.7071) \\
&= \Phi(0.0679) - (1 - \Phi(0.7071)) \\
&= 0.528 - (1 - .761) \\
&= 0.289
\end{aligned}$$

2. Poker hands

a.

$$\binom{49}{2} = 1176$$

b. One pair?

$$\frac{\binom{44}{2} - 11\binom{4}{2}}{\binom{49}{2}} = 0.748$$

where the subtraction is to discount the possibility of drawing a new pair, which would boost the hand to two pairs.

c. Two pair?

$$\frac{\binom{3}{1}\binom{44}{1} + 11\binom{4}{2}}{\binom{49}{2}} = 0.168$$

where the first term on top accounts for drawing another jack, and the other term on top for drawing a new pair. Remeber, drawing a 3 would bump the hand up in status.

d. Three of a kind?

$$\frac{\binom{2}{1}\binom{44}{1}}{\binom{49}{2}} = 0.075$$

where the number on top is the number of ways to draw one three and one non-three/jack. Drawing two jacks would yield a full house.

3. Median Absolute Deviation

a.

$$\begin{aligned}
E[|X - \mu|] &= \int_{-a}^{a} |x| \cdot \frac{1}{2a}\, dx \\
&= 2\int_0^a \frac{x}{2a}\, dx \\
&= \frac{1}{a}\int_0^a x\, dx \\
&= \frac{1}{a}\left[\frac{1}{2}x^2\right]_0^a \\
&= \frac{a}{2}.
\end{aligned}$$

By comparison, the standard deviation is

$$\sqrt{\frac{(2a)^2}{12}} = \frac{a}{\sqrt{3}} > \text{MAD}(X).$$

b.

$$\begin{aligned} E[|X-\mu|] &= \int_{-\infty}^{\infty} |x| f(x)\, dx \\ &= 2\int_0^{\infty} x f(x)\, dx \\ &= \frac{2}{\sqrt{2\pi\sigma^2}} \int_0^{\infty} x e^{-\frac{x^2}{2\sigma^2}}\, dx \\ &= \frac{2}{\sqrt{2\pi\sigma^2}} \left[-\sigma^2 e^{-\frac{x^2}{2\sigma^2}} \right]_0^{\infty} \\ &= \sqrt{\frac{2}{\pi}}\sigma < \sigma = \text{STD}(X) \end{aligned}$$

4. MGF

a.

$$\begin{aligned} M(t) &= \sum_{x=1}^{\infty} e^{xt} \cdot \frac{1}{\ln 2} \cdot \frac{1}{x 2^x} \\ &= \frac{1}{\ln 2} \sum_{x=1}^{\infty} \frac{1}{x} \left(\frac{e^t}{2} \right)^x \\ &= -\frac{1}{\ln 2} \ln \left(1 - \frac{e^t}{2} \right), \end{aligned}$$

for $t < \ln 2$.

b.

$$\mu = M'(0) = \frac{1}{\ln 2}$$

and

$$\sigma^2 = M''(0) - \mu^2 = \frac{2}{\ln 2} - \left(\frac{1}{\ln 2} \right)^2$$