STAT/ELEC 331 HW 9

Problems in addition to those from the book

1. The moment generating function (MGF) can be used to find the moments of a distribution. But it can also be used to determine the distribution of a function of a random variable/vector. This is because the MGF uniquely determines the distribution of a random variable (a fact we did not prove). We used this fact when showing that a linear transformation of a Gaussian was again Gaussian. In this problem we'll apply the MGF to the Poisson distribution.

a. Let $X_1, \ldots, X_n$ be a random sample from a Poi(λ) distribution. Using the MGF, identify the distribution of $X = \sum_{i=1}^{n} X_i$.

b. Using the Central Limit Theorem and the previous result, explain how to approximate a Poi(λ) distribution using the normal distribution. **HINT:** Determine or approximate the distribution of the sum of n independent Poi(λ/n) variates in two different ways.

c. Suppose $X \sim$ Poi(24.9). Use the result of b. and the table for the standard normal CDF to approximate the probability $P(X \geq 30)$.

d. **Extra credit**: (1 point) Use a computer or calculator to compute the probability in c. exactly. How close is the approximation?

2. Sampling uniformly in a sphere

In the problem we'll explore two ways of generating points uniformly at random inside the unit sphere in 3 dimensions, $\{(x_1, x_2, x_3) : x_1^2 + x_2^2 + x_3^2 \leq 1\}$.

a. Briefly describe a method based on rejection sampling. What is the expected number of iterations of the rejection algorithm needed to obtain one sample in the unit sphere?

b. If you also have access to a normal random number generator, an alterative simulation method is possible. To generalize the method discussed in class we need a way to similate a point on the surface of the sphere. For this we can take $\mathbf{W} = \mathbf{Z}/|\mathbf{Z}|$, where $\mathbf{Z}$ is a vector whose coordinates are independent standard normals (you don't need to show this). Using this fact, extend the method discussed in class. In particular, find a way to generate R such that $R\mathbf{W}$ has the desired distribution.

c. Using the second method of simulation (from b.), estimate the probability of the event $\sum_{i=1}^{3} |X_i|^{2/3} \leq 1$. In MATLAB, the following commands are useful: `rand, randn`.

d. Imagine extending this procedure to d dimensions. Go online and find the formula for the volume of a sphere in d dimensions. What happens to the ratio of the volume of the unit sphere to the volume of the circumscribed cube as a function of d? (It might help to plot the ratio). Which of the two methods of simulation seems preferable for large d from a computational point of view?

3. Let X be a random variable and assume the CDF of X is continuous and strictly increasing (so that it has an inverse). Show that $F(X) \sim$ unif$[0, 1]$.

4. Suppose I want to generate samples from a truncated normal as defined in homework 6. One appealing approach is to generate samples from a $N(0, \sigma^2)$ and keep the ones whose absolute value is less that or equal to γ. Is this a valid approach? Why or why not?