

Stat310 HW5 Solutions (100')

(5') 3.5-2 Here $x = \sqrt{y}$, $D_y(x) = 1/2\sqrt{y}$ and $0 < x < \infty$ maps onto $0 < y < \infty$. Thus

$$g(y) = \sqrt{y} \left| \frac{1}{2\sqrt{y}} \right| = \frac{1}{2} e^{-y/2}, \quad 0 < y < \infty.$$

(10') 3.5-6 It is easier to note that

$$\frac{dy}{dx} = \frac{e^{-x}}{(1+e^{-x})^2} \quad \text{and} \quad \frac{dx}{dy} = \frac{(1+e^{-x})^2}{e^{-x}}.$$

Say the solution of x in terms of y is given by x^* . Then the p.d.f. of Y is

$$g(y) = \frac{e^{-x^*}}{(1+e^{-x^*})^2} \left| \frac{(1+e^{-x^*})^2}{e^{-x^*}} \right| = 1, \quad 0 < y < 1,$$

as $-\infty < x < \infty$ maps onto $0 < y < 1$. Thus Y is $U(0, 1)$.

(10')

3.5-12 $E(X) = \int_{-\infty}^{\infty} \frac{x}{\pi(1+x^2)} dx$

$$= \lim_{a \rightarrow -\infty} \left[\frac{1}{2\pi} \ln(1+x^2) \right]_a^0 + \lim_{b \rightarrow +\infty} \left[\frac{1}{2\pi} \ln(1+x^2) \right]_0^b$$
$$= \frac{1}{2\pi} \left[\lim_{a \rightarrow -\infty} \{-\ln(1+a^2)\} + \lim_{b \rightarrow +\infty} \ln(1+b^2) \right].$$

(5')

4.1-4 $\frac{25!}{7!8!6!4!} (0.30)^7 (0.40)^8 (0.20)^6 (0.10)^4 = 0.00405.$

(10')

4.1-8 (a) $P(0 \leq X \leq \frac{1}{2}) = \int_0^{\frac{1}{2}} \int_{x^2}^1 \frac{3}{2} dy dx$

$$= \int_0^{\frac{1}{2}} \frac{3}{2} (1-x^2) dx = \frac{11}{16};$$

(b) $P(\frac{1}{2} \leq Y \leq 1) = \int_{\frac{1}{2}}^1 \int_0^{\sqrt{y}} \frac{3}{2} dx dy$

$$= \int_{\frac{1}{2}}^1 \frac{3}{2} \sqrt{y} dy = 1 - \left(\frac{1}{2}\right)^{3/2};$$

$$\begin{aligned} \text{(c)} \quad P\left(\frac{1}{2} \leq X \leq 1, \frac{1}{2} \leq Y \leq 1\right) &= \int_{\frac{1}{2}}^1 \int_{\frac{1}{2}}^{\sqrt{y}} \frac{3}{2} dx dy \\ &= \int_{\frac{1}{2}}^1 \frac{3}{2} \left(\sqrt{y} - \frac{1}{2}\right) dy \\ &= \frac{5}{8} - \left(\frac{1}{2}\right)^{3/2}; \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad P(X \geq \frac{1}{2}, Y \geq \frac{1}{2}) &= P\left(\frac{1}{2} \leq X \leq 1, \frac{1}{2} \leq Y \leq 1\right) \\ &= \frac{5}{8} - \left(\frac{1}{2}\right)^{3/2}. \end{aligned}$$

(e) X and Y are dependent.

(10')

4.1-12 The area of the space is

$$\int_2^6 \int_1^{14-2t_2} dt_1 dt_2 = \int_2^6 (13 - 2t_2) dt_2 = 20;$$

Thus

$$\begin{aligned} P(T_1 + T_2 > 10) &= \int_2^4 \int_{10-t_2}^{14-2t_2} \frac{1}{20} dt_1 dt_2 \\ &= \int_2^4 \frac{4-t_2}{20} dt_2 \\ &= \left[-\frac{(4-t_2)^2}{40} \right]_2^4 = \frac{1}{10}. \end{aligned}$$

(10')

4.2-6 Note that X is $b(3, 1/6)$, Y is $b(3, 1/2)$ so

(a) $E(X) = 3(1/6) = 1/2,$

(b) $E(Y) = 3(1/2) = 3/2,$

(c) $\text{Var}(X) = 3(1/6)(5/6) = 5/12,$

(d) $\text{Var}(Y) = 3(1/2)(1/2) = 3/4;$

(e) $\text{Cov}(X, Y) = 0 + (1)f(1, 1) + 2f(1, 2) + 2f(2, 1) - (1/2)(3/2)$
 $= (1)(1/6) + 2(1/8) + 2(1/24) - 3/4$
 $= -1/4;$

(f) $\rho = \frac{-1/4}{\sqrt{\frac{5}{12} \cdot \frac{3}{4}}} = \frac{-1}{\sqrt{5}}.$

(10')

4.2-12 (a) $f_1(x) = \int_x^1 8xy \, dy = 4x(1 - x^2), \quad 0 \leq x \leq 1,$

$$f_2(y) = \int_0^y 8xy \, dx = 4y^3, \quad 0 \leq y \leq 1;$$

(b) $\mu_x = \int_0^1 x4x(1 - x^2) \, dx = \frac{8}{15},$

$$\mu_y = \int (y * 4y^3 \, dy = \frac{4}{5},$$

$$\sigma_x^2 = \int_0^1 (x - 8/15)^2 4x(1 - x^2) \, dx = \frac{11}{225},$$

$$\sigma_y^2 = \int ((y - 4/5)^2 * 4y^3 \, dy = \frac{2}{75},$$

$$\text{Cov}(X, Y) = \int_0^1 \int_x^1 (x - 8/15)(y - 4/5)8xy \, dy \, dx = \frac{4}{225},$$

$$\rho = \frac{4/225}{\sqrt{(11/225)(2/75)}} = \frac{2\sqrt{66}}{33};$$

(c) $y = \frac{20}{33} + \frac{4x}{11}.$

(10')

- 4.3-4 (a) X is $b(400, 0.75)$;
(b) $E(X) = 300$, $\text{Var}(X) = 75$;
(c) $b(300, 2/3)$;
(d) $E(Y) = 200$, $\text{Var}(Y) = 200/3$.
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(10')

- 4.3-8 (a) X and Y have a trinomial distribution with $n = 30$, $p_1 = 1/6$, $p_2 = 1/6$.
(b) The conditional p.d.f. of X , given $Y = y$, is

$$b\left(n - y, \frac{p_1}{1 - p_2}\right) = b(30 - y, 1/5).$$

- (c) Since $E(X) = 5$ and $\text{Var}(X) = 25/6$, $E(X^2) = \text{Var}(X) + [E(X)]^2 = 25/6 + 25 = 175/6$. Similarly, $E(Y) = 5$, $\text{Var}(Y) = 25/6$, $E(Y^2) = 175/6$. The correlation coefficient is

$$\rho = -\sqrt{\frac{(1/6)(1/6)}{(5/6)(5/6)}} = -1/5$$

so

$$E(XY) = -1/5\sqrt{(25/6)(25/6)} + (5)(5) = 145/6.$$

Thus

$$E(X^2 - 4XY + 3Y^2) = \frac{175}{6} - 4\left(\frac{145}{6}\right) + 3\left(\frac{175}{6}\right) = \frac{120}{6} = 20.$$

(10')

- 4.3-16 (a) $h(y|x) = \frac{1}{x}$, $0 < y < x$, $0 < x < 1$,
(b) $E(Y|x) = \int_0^x \frac{y}{x} dy = \frac{x}{2}$,
(c) $f(x, y) = h(y|x)f_1(x) = \left(\frac{1}{x}\right)(1) = \frac{1}{x}$, $0 < y < x$, $0 < x < 1$,
(d) $f_2(y) = \int_y^1 \frac{1}{x} dx = -\ln y$, $0 < y < 1$.