1 Solution to Problem 2.1

I incorrectly worked this exercise instead of 2.2, so I decided to include the solution anyway.

(a) We have $X = Y^{1/3}$, which is a 1-1 function. It maps the interval $(0, 1)$ (where $X$ lives) onto itself, so the support of (the distribution) of $Y$ is $(0, 1)$.

$$f_Y(y) = f_X(y^{1/3}) \left| \frac{dy}{dy} y^{1/3} \right|$$

$$= 42y^{5/3} \left(1 - y^{1/3}\right) \frac{1}{3} y^{-2/3}$$

$$= 14y \left(1 - y^{1/3}\right), \quad 0 < y < 1.$$  

Checking the integral,

$$\int_0^1 14y \left(1 - y^{1/3}\right) \, dy$$

$$= 14 \int_0^1 \left(y - y^{4/3}\right) \, dy$$

$$= 14 \left[ \frac{1}{2} y^2 - \frac{3}{7} y^{7/3} \right]_{y=0}$$

$$= 14 \left[ \frac{1}{2} - \frac{3}{7} \right]$$

$$= 14 \left[ \frac{7}{14} - \frac{6}{14} \right]$$

$$= 14 \left[ \frac{1}{14} \right]$$

$$= 1.$$

(b) $X = (Y - 3)/4$ is 1-1. The interval $(0, \infty)$, which is the support of $X$, is mapped to $(3, \infty)$ under $Y = 4X + 3$, so the support of $Y$ is $(3, \infty)$. Computing the pdf for $Y$:

$$f_Y(y) = f_X((y - 3)/4) \left| \frac{dy}{dy} (y - 3)/4 \right|$$

$$= 7 \exp \left(-7(y - 3)/4\right) \frac{1}{4}$$

$$= \frac{7}{4} \exp \left(-7(y - 3)/4\right), \quad y > 3.$$  

Checking the integral,

$$\int_3^\infty \frac{7}{4} \exp \left(-7(y - 3)/4\right) \, dy$$
\[ \int_{0}^{\infty} \frac{7}{4} \exp\left(-\frac{7}{4}x\right) \, dx \quad \text{(substitute } x = y - 3) \]
\[ = 1, \]

since the latter expression is the integral over its entire range of the exponential, mean 4/7, pdf.

(c) The function \( y = g(x) = x^2 \) is 1-1 on \( x \in (0, 1) \), and maps \( (0, 1) \) onto \( (0, 1) \), so the support of \( Y \) is \((0, 1)\). As \( X = Y^{1/2} \),

\[ f_Y(y) = f_X\left(y^{1/2}\right) \left| \frac{d}{dy} y^{1/2} \right| = 30y \left(1 - y^{1/2}\right)^2 \frac{1}{2} y^{-1/2} = 15y^{1/2} \left(1 - y^{1/2}\right)^2, \quad 0 < y < 1. \]

Checking the integral,

\[ \int_{0}^{1} 15y^{1/2} \left(1 - y^{1/2}\right)^2 \, dy \]
\[ = 15 \int_{0}^{1} y^{1/2} \left(1 - 2y^{1/2} + y\right) \, dy \]
\[ = 15 \int_{0}^{1} \left(y^{1/2} - 2y + y^{3/2}\right) \, dy \]
\[ = 15 \left[ \frac{2}{3}y^{3/2} - y^2 + \frac{2}{5}y^{5/2} \right]_{y=0}^{1} \]
\[ = 15 \left[ \frac{2}{3} - 1 + \frac{2}{5} \right] \]
\[ = 15 \left[ \frac{10}{15} - \frac{15}{15} + \frac{6}{15} \right] \]
\[ = 10 - 15 + 6 \]
\[ = 1. \]

2 Solution to Problem 2.2

We apply Theorem 2.1.5 in each case.
(a) $X = Y^{1/2}$ so $dx = (1/2)y^{-1/2}dy$. Note that as $x$ ranges over $(0,1)$, $x^2$ also ranges over $(0,1)$.

$$f_Y(y) = f_X(y^{1/2}) (1/2)y^{-1/2}, \quad 0 < y < 1,$$

$$= (1/2)y^{-1/2}, \quad 0 < y < 1.$$

(b) If $Y = -\log X$, then $X = e^{-Y}$ and $dx = -e^{-y}dy$. As $x$ ranges over $(0,1)$, $y$ ranges over $(0,\infty)$, so we have

$$f_Y(y) = f_X(e^{-y}) e^{-y}, \quad y > 0,$$

$$= \frac{(n+m+1)!}{n!m!} e^{-ny} (1 - e^{-y})^m e^{-y}, \quad y > 0,$$

$$= \frac{(n+m+1)!}{n!m!} e^{-(n+1)y} (1 - e^{-y})^m \quad y > 0.$$

(c) If $Y = e^X$, then $X = \log Y$ and $dx = y^{-1}dy$. Also, as $x$ ranges over $(0,\infty)$, $y$ ranges over $(1,\infty)$. Hence,

$$f_Y(y) = f_X(\log y) y^{-1}, \quad y > 1,$$

$$= \frac{1}{\sigma^2} \log y \exp \left[ -\frac{1}{2\sigma^2} (\log y)^2 \right] y^{-1}, \quad y > 1,$$

$$= \frac{1}{\sigma^2} \log y \exp \left[ -\frac{1}{2\sigma^2} (\log y)^2 \right], \quad y > 1.$$

3 Solution to Problem 2.9

The cdf for $1 < x < 3$ is

$$F(x) = \int_1^x (y-1)/2 \, dy$$

$$= \int_0^{x-1} y/2 \, dy$$

$$= \left[ y^2/4 \right]_{y=0}^{x-1}$$

$$= (x-1)^2/4.$$
Thus, the cdf over the whole real line is

\[
F(x) = \begin{cases} 
0 & \text{if } x < 1, \\
(x - 1)^2/4 & \text{if } 1 \leq x < 3, \\
1 & \text{if } 3 \leq x.
\end{cases}
\]

Note that \( F(x) \) is continuous at \( x = 1 \) as \( (x - 1)^2/4 = 0 \) at \( x = 1 \), and also at \( x = 3 \) as \( (x - 1)^2/4 = 1 \) at \( x = 3 \). Thus,

\[ U = (X - 1)^2/4 \]

has a Unif(0,1) distribution.

4 Solution to Problem 2.11

(a) Find \( E[X^2] \) “directly” must mean integrating against the \( N(0,1) \) pdf:

\[
\int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \left( xe^{-x^2/2} \right) \, dx. \tag{1}
\]

Applying integration by parts:

\[
\int u \, dv = uv - \int v \, du \\
u = x \\
dv = xe^{-x^2/2} \, dx \\
\]

we obtain

\[
\int_{-\infty}^{\infty} x^2 e^{-x^2/2} \, dx = -\left. xe^{-x^2/2} \right|_{x=-\infty}^{\infty} + \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \int_{-\infty}^{\infty} e^{-x^2/2} \, dx = \sqrt{2\pi},
\]

so we obtain \( E[X^2] = 1 \) after we put the \( 1/\sqrt{2\pi} \) factor back in the pdf.
Now, using the pdf of Example 2.1.7: if $Y = X^2$ then

$$f_Y(y) = \frac{1}{2\sqrt{y}} [f_X(\sqrt{y}) + f_X(-\sqrt{y})]$$

$$= \frac{1}{2\sqrt{y}} \frac{1}{\sqrt{2\pi}} e^{-y/2}$$

$$= \frac{y^{-1/2}}{\sqrt{2\pi}} e^{-y/2}, \quad y > 0.$$  

Now,

$$E[Y] = \int_0^\infty y y^{-1/2} \sqrt{2\pi} e^{-y/2} dy$$

$$= \int_0^\infty y^{1/2} \sqrt{2\pi} e^{-y/2} dy$$

$$= -2 \frac{y^{1/2} e^{-y/2}}{\sqrt{2\pi}} \bigg|_y^{\infty} + \int_0^\infty \frac{y^{-1/2}}{\sqrt{2\pi}} e^{-y/2} dy,$$  \hspace{1cm} (2)

where we have used integration by parts with

$$u = y^{1/2} \quad dv = e^{-y/2} dy$$

$$du = \frac{1}{2} y^{-1/2} dy \quad v = -2e^{-y/2}$$

Observe that the first term on the r.h.s. of (2) is 0, since $y^{1/2} = 0$ at $y = 0$ and $\lim_{y \to \infty} y^{1/2} e^{-y/2} = 0$. Also, the second term on the r.h.s. of (2) is 1 since it is the integral of $f_Y(y)$ over its whole range.

(b) The pdf of $Y = |X|$ is similar (but easier) to the derivation in Example 2.1.7. To give the details: assume $y \geq 0$,

$$F_Y(y) = P[Y \leq y]$$

$$= P[|X| \leq y]$$

$$= P[-y \leq X \leq y]$$

$$= P[-y < X \leq y] \quad \text{since } X \text{ is a cont. rv}$$

$$= F_X(y) - F_X(-y).$$
Differentiating, we get

$$f_Y(y) = f_X(y) + f_X(-y)$$
$$= 2f_X(y)$$

since by symmetry $f_X(y) = f_X(-y)$ (recall $X \sim N(0, 1)$). Thus,

$$f_Y(y) = \sqrt{\frac{2}{\pi}} e^{-y^2/2}, \; y > 0.$$

We have

$$E[Y] = \sqrt{\frac{2}{\pi}} \int_0^\infty ye^{-y^2/2} dy$$
$$= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-w} dw \; \text{ substitute } w = y^2/2, \; dw = ydy$$
$$= \sqrt{\frac{2}{\pi}}.$$

We have $E[Y^2] = E[X^2] = 1$, so

$$\text{Var}[Y] = E[Y^2] - (E[Y])^2 = 1 - 2/\pi = 0.3633802.$$

5 Solution to Problem 2.14

(a) There are various approaches to this problem – under certain assumptions one can use integration by parts. However, the easiest approach by far is to use the following trick:

$$1 - F(x) = P[X > x] = \int_x^\infty f(y) \, dy.$$

Plugging this into the expression on the r.h.s. of the proposed formula:

$$\int_0^\infty 1 - F(x) \, dx$$
$$= \int_0^\infty \int_x^\infty f(y) \, dy \, dx$$
$$= \int_0^\infty \int_0^y f(y) \, dx \, dy.$$
In the last expression, we simply switched the order of integration. Evaluating the double integral can be done by iterated integration either way: first over \( x \) or first over \( y \). Note that we are integrating (in the plane \( \mathbb{R}^2 \)) over the region \( \{(x, y) : 0 \leq x \leq y < \infty \} \). Since \( f(y) \) is a constant when integrating with respect to \( x \), the last integral is

\[
\int_0^\infty y f(y) \, dy = E[X],
\]

since \( f(\cdot) \) is the pdf for \( X \).

(b) A similar argument works here:

\[
1 - F(k) = \sum_{j=k+1}^{\infty} f(j),
\]

so

\[
\sum_{k=0}^{\infty} [1 - F(k)]
\]

\[
= \sum_{k=0}^{\infty} \sum_{j=k+1}^{\infty} f(j)
\]

\[
= \sum_{j=0}^{\infty} \sum_{k=0}^{j-1} f(j)
\]

\[
= \sum_{j=0}^{\infty} j f(j)
\]

\[
= E[X].
\]

6 Solution to Problem 2.22

I guess I had too much spare time and prepared a solution for this exercise as well.

(a) Clear \( f(x) \geq 0 \), as \( \beta > 0 \), \( x^2 \geq 0 \), and \( e^{-x^2/2} > 0 \). Checking that it integrates to 1,

\[
\int_0^\infty \frac{4}{\beta^3 \sqrt{\pi}} x^2 e^{-x^2/\beta^2} \, dx
\]

\[
= \int_0^\infty \sqrt{\frac{2}{\pi}} y^2 e^{-y^2/2} \, dy
\]

\[
= 2 \int_0^\infty \frac{1}{\sqrt{2\pi}} y^2 e^{-y^2/2} \, dy
\]

(3)
\[
\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} y^2 e^{-y^2/2} dy \quad (4) \\
= 1. \quad (5)
\]
where at (3), we substituted \( y = \sqrt{2}x/\beta \), at (4) we used the symmetry of the \( N(0,1) \) pdf, and at (5) we used the fact that \( E[Y^2] = 1 \) when \( Y \sim N(0,1) \), something we already proved back in Exercise 2.11.

(b) Applying the same substitution as at (3), we have

\[
E[X] = \int_{0}^{\infty} \frac{4}{\beta^3 \sqrt{2\pi}} x^3 e^{-x^2/\beta^2} \, dx
\]
\[
= \beta \int_{0}^{\infty} \sqrt{\frac{1}{\pi}} y^3 e^{-y^2/2} \, dy
\]
\[
= -\frac{\beta}{\sqrt{\pi}} y^2 e^{-y^2/2} \bigg|_{y=0}^{\infty} + 2\beta \frac{\sqrt{\pi}}{\sqrt{\pi}} \int_{0}^{\infty} ye^{-y^2/2} \, dy
\]
\[
= \frac{2\beta}{\sqrt{\pi}} \int_{0}^{\infty} e^{-w} \, dw, \quad (7)
\]
\[
= \frac{2\beta}{\sqrt{\pi}}. \quad (8)
\]
where at (6) we did an integration by parts with

\[
u = y^2 \quad dv = ye^{-y^2/2} \, dy
\]
\[
du = 2y \, dy \quad v = -e^{-y^2/2},
\]
and at (7) we substituted \( w = y^2/2 \). Proceeding on to the next moment,

\[
E[X^2] = \int_{0}^{\infty} \frac{4}{\beta^3 \sqrt{2\pi}} x^4 e^{-x^2/\beta^2} \, dx
\]
\[
= \frac{\beta^2}{\sqrt{2\pi}} \int_{0}^{\infty} y^4 e^{-y^2/2} \, dy
\]
\[
= -\frac{\beta^2}{\sqrt{2\pi}} y^3 e^{-y^2/2} \bigg|_{y=0}^{\infty} + \frac{3\beta^2}{\sqrt{2\pi}} \int_{0}^{\infty} y^2 e^{-y^2/2} \, dy
\]
\[
= \frac{3\beta^2}{2} \int_{-\infty}^{\infty} \frac{y^2}{\sqrt{2\pi}} e^{-y^2/2} \, dy
\]
\[
= \frac{3\beta^2}{2}. \quad (8)
\]
Our integration by parts here was
\[ u = y^3 \quad dv = ye^{-y^2/2} \, dy \]
\[ du = 3y^2 \, dy \quad v = -e^{-y^2/2} \]

Thus, we have
\[ \text{Var}[X] = \frac{3\beta^2}{2} - \left( \frac{2\beta}{\sqrt{\pi}} \right)^2 = \left[ \frac{3}{2} - \frac{4}{\pi} \right] \beta^2 = 0.2267605\beta^2. \]

### 7 Solution to Problem 2.32

We have by the chain rule for differentiation,
\[ \frac{d}{dt} \log M(t) = \frac{1}{M(t)} \frac{d}{dt} M(t). \]

Since \( M(0) = 1 \), we obtain that \( S'(0) = M'(0)/M(0) = M'(0) = E[X] \). For the second derivative, we apply the rule for differentiating a quotient:
\[ S^{(2)}(t) = \frac{d}{dt} \frac{M'(t)}{M(t)} = \frac{M(t)M''(t) - (M'(t))^2}{M^2(t)}, \]

so
\[ S^{(2)}(0) = \frac{M(0)M''(0) - (M'(0))^2}{M^2(0)} = M''(0) - (M'(0))^2 = E[X^2] - (E[X])^2 = \text{Var}[X]. \]

### 8 Solution to Problem 2.33

(a) Mgf for the Poisson(\( \lambda \)):
\[ M(t) = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k}{k!} e^{-\lambda} \]
\[ = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\lambda e^t)^k}{k!} \]
\[ = e^{-\lambda} \exp \left[ \lambda e^t \right] \]
\[ = \exp \left[ \lambda (e^t - 1) \right]. \]
At (9), we used the formula for the Taylor series for the exponential function. Since this converges everywhere, there are no restrictions on \( t \), i.e. the mgf is finite everywhere.

Using the trick from the previous problem, we have
\[
E[X] = S'(0) = \frac{d}{dt} \lambda \left( e^t - 1 \right) \bigg|_{t=0} = \lambda e^t \bigg|_{t=0} = \lambda,
\]
and
\[
\text{Var}[X] = S''(0) = \frac{d}{dt} \lambda e^t \bigg|_{t=0} = \lambda.
\]

(b) Mgf for the Geometric(\( p \)):
\[
M(t) = \sum_{k=0}^{\infty} e^{kt} (1-p)^k p = p \sum_{k=0}^{\infty} (e^t (1-p))^k = \frac{p}{1 - e^t(1-p)}, \tag{10}
\]
where we used the formula for summing a geometric series:
\[
\sum_{k=0}^{\infty} r^k = \frac{1}{1-r}, \quad |r| < 1.
\]
The region for convergence \( |r| < 1 \) translates into
\[
e^t(1-p) < 1 \implies t < -\log(1-p).
\]
Note that \(-\log(1-p) > 0\) for \( 0 < p < 1 \), so the mgf is finite in a neighborhood of 0.

Computing the moments:
\[
S'(t) = \frac{d}{dt} \left[ \log p - \log \left[ 1 - e^t(1-p) \right] \right] = e^t(1-p) / \left[ 1 - e^t(1-p) \right].
\]
Substituting in \( t = 0 \) gives
\[
E[X] = (1-p) / [1 - (1-p)] = (1-p)/p.
\]
Also,
\[
S''(t) = \frac{d}{dt} e^t(1-p) / \left[ 1 - e^t(1-p) \right] = \frac{d}{dt} (1-p) / \left[ e^{-t} - (1-p) \right] = (1-p)e^{-t} / \left[ e^{-t} - (1-p) \right]^2,
\]
and so evaluating this at \( t = 0 \) gives

\[
\text{Var}[X] = (1 - p)/p^2.
\]

(c) As noted in lecture, it is much easier to start with the \( N(0, 1) \) case and then extend to \( N(\mu, \sigma^2) \) using transformation formulae as in Theorem 2.3.15. So, let \( Z \sim N(0, 1) \). Then

\[
M_Z(t) = \int_{-\infty}^{\infty} e^{tz} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz
= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \exp \left[ -\frac{1}{2} (z - t)^2 + t^2/2 \right] \, dz
= e^{t^2/2}.
\]

Now, \( X = \sigma Z + \mu \) is \( N(\mu, \sigma^2) \), and

\[
M_X(t) = E[e^{tX}] = E[\exp(t\sigma Z + t\mu)] = \exp(\mu t) E[\exp(t\sigma Z)]
= \exp(\mu t) M_Z(\sigma t) = \exp(\mu t) \exp(\sigma^2 t^2/2) = \exp(\mu t + \sigma^2 t^2/2),
\]

as desired.

Now the mean and variance computations are especially easy with the trick from Exercise 2.32:

\[
E[X] = \frac{d}{dt} \left( \mu t + \sigma^2 t^2/2 \right) \bigg|_{t=0} = \left( \mu + \sigma^2 t \right) \bigg|_{t=0} = \mu,
\]

and

\[
\text{Var}[X] = \frac{d}{dt} \left( \mu + \sigma^2 t \right) \bigg|_{t=0} = \sigma^2.
\]