Solutions to Homework 1

September 21, 2017

Solution to Exercise 1.1.15

(i) Clearly $F(x) \geq 0$ for all $x \in I\!R$, so from the definition,

$$F^-(0) \;=\; \inf\{\, x \,:\, F(x) \geq 0 \,\} \;=\; \inf I\!R \;=\; -\infty.$$

Of course, this is true for any c.d.f. We see from the picture that

$$\{x \in I\!R : F(x) \geq 0\} \;=\; \{x \in I\!R : F(x) = 0\} \;=\; (-\infty, 0],$$

so

$$F^+(x) \;=\; \sup(-\infty, 0] \;=\; 0.$$

(ii) It appears from Figure 1.2 that $F(x)$ is strictly increasing and continuous for $x \in (0,1)$, so F^{-1} exists. Further, $F(0) = 0$ and $F(1) = 1/2$. So for $\alpha \in (0, 1/2)$,

$$\begin{aligned} \{x : F(x) \geq \alpha\} &\;=\; [F^{-1}(\alpha), \infty), \\ \{x : F(x) \leq \alpha\} &\;=\; (-\infty, F^{-1}(\alpha)]. \end{aligned}$$

Taking sup and inf, respectively, we get $F^{-1}(\alpha)$ in both cases, as claimed. Note that it appears that the function is linear in this range, so

$$F(x) \;=\; x/2, \quad 0 \leq x < 1,$$

and thus

$$F^-(\alpha) \;=\; F^+(\alpha) \;=\; F^{-1}(\alpha) \;=\; 2\alpha, \quad 0 < \alpha < 1/2.$$

(iii) Clearly $F(1-) = \lim_{x\uparrow 1} F(x) = 1/2$ and $F(1) = 3/4$. Thus, if $1/2 \leq \alpha < 3/4$,

$$\begin{aligned} \{x : F(x) \geq \alpha\} &\;=\; [1, \infty), \\ \{x : F(x) \leq \alpha\} &\;=\; (-\infty, 1). \end{aligned}$$

Thus, we see

$$F^-(\alpha) \;=\; F^+(\alpha) \;=\; 1, \quad 1/2 \leq \alpha < 3/4.$$

However, $F(x) < 1/2$ for $x < 1$ and $F(x) \geq 3/4$ for $x \geq 1$, so there is no value of x such that $F(x) = \alpha$ when $1/2 \leq \alpha < 3/4$.

(iv) It is stated that F is constant on the interval $[1, 2]$, and it seems clear from the picture that the value is $3/4$. Therefore,

$$\begin{aligned} \{x : F(x) \geq 3/4\} &= [1, \infty), \\ \{x : F(x) \leq 3/4\} &= (-\infty, 2]. \end{aligned}$$

Taking inf and sup respectively gives

$$\begin{aligned} F^{-}(3/4) &= 1, \\ F^{+}(3/4) &= 2. \end{aligned}$$

Note that these are the smallest and largest values of x such that $F(x) = 3/4$. If one is forced to assign a unique value to the 75'th percentile, I suppose you can average these two values (so it's 1.5), or maybe go 3/4 of the way between then (so it's 1.75).

(v) It's not stated in the text, but it looks like $F(x)$ is strictly increasing and continuous for $2 < x < 3.1$. (I say "looks like" because there could be little jumps and flat spots that are so small that they don't show up on the graph at the scale it is plotted.) Then, $F^{-1}(\alpha)$ is defined for $F(2) = 3/4 < \alpha < F(3.1) = 1$ and the same argument as in (ii) above applies.

(vi) Clearly,

$$\begin{aligned} \{x : F(x) \geq 1\} &= [3.1, \infty), \\ \{x : F(x) \leq 1\} &= (-\infty, \infty). \end{aligned}$$

The latter equation holds for all c.d.f.'s, of course. Taking inf and sup, respectively, gives

$$\begin{aligned} F^{-}(1) &= 3.1, \\ F^{+}(1) &= \infty. \end{aligned}$$

Solution to Exercise 1.1.16 (i) If you try proving this directly, you will probably go around in circles for awhile. Whenever that happens, remember to try an indirect proof - in this case, a proof by contradiction. Assume $F^{-}(\alpha) > F^{+}(\alpha)$, and derive a contradiction. Now by definition,

$$F^{-}(\alpha) = \inf\{x : F(x) \geq \alpha\},$$

and inf is "greatest lower bound." Since we are assuming $F^+(\alpha) < F^-(\alpha)$, it follows that we can find a value x_0 such that

$$F^+(\alpha) \ < \ x_0 \ < \ F^-(\alpha).$$

Then by definition of "greatest lower bound" we must have $F(x_0) < \alpha$. However,

$$F^+(\alpha) \ = \ \sup\{x : F(x) \leq \alpha\},$$

Put otherwise, $F^+(\alpha)$ is the "least upper bound" of the set on the right hand side of this equation. Note that x_0 is an element of this set, and that contradicts $F^+(\alpha)$ being an upper bound. This is the contradiction that proves the result.

(ii) It is pretty easy to see that this is incorrect. Again, let's look at the negation and assume $F^-(\alpha) < F^+(\alpha)$, which is the only possibility when the two are not equal because of part (i). So as above pick x_0 such that

$$F^-(\alpha) \ < \ x_0 \ < \ F^+(\alpha).$$

Then by the definition of $F^-(\alpha)$, we must have $F(x_0) \geq \alpha$, and by the definition of $F^+(\alpha)$ we must have $F(x_0) \leq \alpha$, so we see that if $F^-(\alpha) \neq F^+(\alpha)$, then there is an nonempty open interval of x values where $F(x) = \alpha$, namely $(F^-(\alpha), F^+(\alpha))$. Note that this is a subset of $F^{-1}(\{\alpha\})$. There are only two other possibilities besides $F^{-1}(\{\alpha\})$ containing an opern interval, namely, an empty set or a singleton set. This follows because if there are two distinct points $x_1, x_2 \in F^{-1}(\{\alpha\})$, then the interval $(x_1, x_2) \subset F^{-1}(\{\alpha\})$. Thus, if $F^-(\alpha) \neq F^+(\alpha)$, i.e., if $F^-(\alpha) < F^+(\alpha)$, then $F^{-1}(\{\alpha\})$ has at least two elements (in fact, infinitely many elements). Thus, by taking the contrapositive, if $F^{-1}(\{\alpha\})$ has less than two elements (is either a singleton or the empty set), then $F^-(\alpha) = F^+(\alpha)$.

Lets see if we can prove the converse. Assume $F^-(\alpha) = F^+(\alpha)$, and let x_0 be their common value. It follows from the definition of F^- that $\forall x > x_0$, $F(x) \geq \alpha$. In fact, by the argument above, we must have $\forall x > x_0$, $F(x) > \alpha$, for if there were an $x > x_0$ with $F(x) = \alpha$, then by definition of F^+ we would have $F^+(\alpha) \geq x > x_0$ contradicting the assumption $F^+(\alpha) = x_0$. Similarly $\forall x < x_0$, $F(x) < \alpha$. Thus, the only possible point in $F^{-1}(\{\alpha\})$ is x_0, so $F^{-1}(\{\alpha\})$ is either empty or a singleton set.

Thus, a necessary and sufficient condition that $F^-(\alpha) = F^+(\alpha)$ is that $F^{-1}(\{\alpha\})$ is either empty or a singleton set. If $F^-(\alpha)$ is a point of increase, it is easy to show that $F^{-1}(\{\alpha\})$ is either empty or a singleton set. However, the converse is not true, that is, if $F^{-1}(\{\alpha\})$ is either empty or a singleton set, then it is not necessarily true that $F^-(\alpha)$ is a point of increase.

We can characterize the situation where $F^{-1}(\{\alpha\})$ is either empty or a singleton set a little more completely. If $F^{-1}(\{\alpha\}) = \emptyset$, then of course there is no $x \in I\!\!R$ with $F(x) = \alpha$. For $\alpha \in (0,1)$, this means $x_0 = F^-(\alpha) = F^+(\alpha)$ is a point where F is discontinuous, i.e., a point with positive probability mass. If $F^{-1}(\{\alpha\}) = \{x_0\}$, then $F(x_0) = \alpha$, $x_0 = F^-(\alpha) = F^+(\alpha)$. Also, in this case, for $x < x_0$ we have $F(x) < \alpha$ by definition of $F^+(\alpha)$, and for $x > x_0$ we have $F(x) > \alpha$ by definition of $F^-(\alpha)$.

(iii) Suppose F is continuous and strictly increasing, then the inverse function $F^{-1} : (0,1) \longrightarrow I\!\!R$ exists. Thus $F^{-1}(\{\alpha\}) = \{F^{-1}(\alpha)\}$ and hence $F^-(\alpha) = F^+(\alpha)$.

(iv) $F^-(0) = \inf\{x : F(x) \geq 0\} = \inf I\!\!R = -\infty$, and similarly for $F^+(1)$.

(v) Since $F^-(\alpha)$ is the greatest lower bound of $\{x : F(x) \geq \alpha\}$, then for any $\epsilon > 0$ that $y = F^-(\alpha) - \epsilon$ cannot be in the set $\{x : F(x) \geq \alpha\}$, which implies $F(F^-(\alpha) - \epsilon) < \alpha$. Thus,

$$F(F^-(\alpha) - 0) \; = \; \lim_{\epsilon \downarrow 0} F(F^-(\alpha) - \epsilon) \; \leq \; \alpha.$$

To prove the other result, we proceed by contradiction. Suppose $F(F^-(\alpha)) < \alpha$. Then, since F is right continuous, there would exist an $x > F^-(\alpha)$ such that $F(x) < \alpha$. But then this x is also a lower bound of $\{x : F(x) \geq \alpha\}$ but it is greater than the greatest lower bound (namely $F^-(\alpha)$), which gives the contradiction.

Moving on to the results about F^+, if $\epsilon > 0$ then $F^+(\alpha) - \epsilon$ must be in $\{x : F(x) \leq \alpha\}$ since $F^+(\alpha)$ is the least upper bound of this set, which is an interval, as noted above. Thus, $F(F^+(\alpha) - \epsilon) \leq \alpha$. Taking limit as $\epsilon \downarrow 0$ shows $F(F^+(\alpha) - 0) \leq \alpha$. The other result follows by a contradiction as before: if $F(F^+(\alpha)) < \alpha$, then by right continuity there is an $x > F(F^+(\alpha))$ with $F(x) < \alpha$, but then this x is in the set $\{x : F(x) \leq \alpha\}$ contradicting the fact that $F^+(\alpha)$ is an upper bound for this set.

(vi) Clearly x is in both the sets

$$\begin{aligned} L &= \{y : F(y) \geq F(x)\} \\ U &= \{y : F(y) \leq F(x)\}. \end{aligned}$$

As $F^-(F(x))$ is a lower bound of L, it follows that $F^-(F(x)) \leq x$. Since $F^+(F(x))$ is an upper bound of U, it follows that $F^+(F(x)) \geq x$.

Solution to Exercise 1.1.17 **(a)** If y_i is a unique (non-replicated) value, then $y_{i-1} < y_i < y_{i+1}$ (where one of the inequalities is not present of $i = 1$ or $i = n$). We then have that

$$\hat{F}_n(x) \;=\; i/n, \quad y_i \le x < y_{i+1}$$

provided $i < n$, and if $i = n$ then

$$\hat{F}_n(x) \;=\; 1, \quad y_n \le x < \infty.$$

Clearly, when y_i is not a replicate then we have

$$\begin{aligned} \hat{F}_n^-(i/n) &= \inf\{\, x \,:\, \hat{F}_n(x) \ge i/n \,\} \\ &= \inf\,[y_i, \infty) \\ &= y_i. \end{aligned}$$

Clearly this holds for $1 \le i \le n$, but does not hold for $i = 0$. Also,

$$\begin{aligned} \hat{F}_n^+(i/n) &= \sup\{\, x \,:\, \hat{F}_n(x) \le i/n \,\} \\ &= \sup\,(-\infty, y_{i+1}) \\ &= y_{i+1}. \end{aligned}$$

Clearly this holds for $0 \le i < n$, but does not hold for $i = n$ unless we define $y_{n+1} = \infty$.

If there are replicated values, things get more complicated. Suppose $j \le i \le k$ are such that $y_{j-1} < y_j = y_{j+1} = \cdots = y_i = \cdots y_k < y_{k+1}$ so the total number of replicates is $k - j + 1$. Then

$$\hat{F}_n(x) \;=\; k/n, \quad y_j \le x < y_{k+1}$$

where the upper limit on x will be replaced with ∞ if $k = n$. If $i < k$, then there is more than 1 replicate, and we will have that there is no value of x such that $\hat{F}_n(x) = i/n$ since $\hat{F}_n(y_j - 0) = (j-1)/n < i/n < k/n$. In this case,

$$\{x : \hat{F}_n(x) \ge i/n\} \;=\; [y_j, \infty),$$

and we get

$$\hat{F}_n^-(i/n) \;=\; y_j = y_i.$$

If $i = k$, then we have $\hat{F}_n(x) = i/n$ for the range of x's above, but still

$$\{x : \hat{F}_n(x) \ge i/n\} \;=\; [y_j, \infty),$$

and the result for $\hat{F}_n^-(i/n)$ is the same. Thus these formulae hold for $1 \le j < k \le n$ and $j \le i \le k$ but not for $i = 0$ unless we define $y_0 = -\infty$.

Still assuming replicated values as in the last paragraph, if $j \le i < k$ then

$$\{x : \hat{F}_n(x) \le i/n\} \;=\; (-\infty, y_j),$$

and

$$\hat{F}_n^+(i/n) \;=\; y_j = y_i = y_{i+1} = y_k.$$

Note that in this case for $x < y_j$ we have $\hat{F}_n(x) \le (i-1)/n$ and for $x \ge y_j$ we have $\hat{F}_n(x) \ge k/n > i/n$. Now when $i = k$ we have

$$\{x : \hat{F}_n(x) \le k/n\} \;=\; (-\infty, y_{k+1})$$

and hence

$$\hat{F}_n^+(k/n) \;=\; y_{k+1}$$

as claimed. The formulae hold as long as $0 \le i < n$.

(b) If n is odd, then $(n+1)/2$ is an integer. In this case there is no value of x such that $\hat{F}_n(x) = 1/2$. Note that $\hat{F}_n(y_{(n+1)/2} - 0) = 1/2 - 1/(2n)$ and $\hat{F}_n(y_{(n+1)/2}) = 1/2 + 1/(2n)$. Clearly

$$\begin{aligned} \{x : \hat{F}_n(x) \le 1/2\} &= (-\infty, y_{(n+1)/2}), \\ \{x : \hat{F}_n(x) \ge 1/2\} &= [y_{(n+1)/2}, \infty). \end{aligned}$$

Taking sup and inf respectively gives

$$\hat{F}_n^-(1/2) \;=\; y_{(n+1)/2}, \quad \hat{F}_n^+(1/2) \;=\; y_{(n+1)/2}.$$

Since the two values are equal, we get that they are both equal to $\mathrm{M}ed(\hat{F}_n)$.

Now consider the case n is even. If $y_{n/2} < y_{n/2+1}$ then we get that

$$[y_{n/2}, y_{n/2+1}) \subset \{x : \hat{F}_n(x) \;=\; 1/2\}.$$

Note that if there are replicated values with possibly $y_{n/2} = y_{n/2+1}$ then there will be some maximal k such that $y_k = y_{n/2}$ and some minimal j with $y_j = y_{n/2}$ and we will have

$$[y_{j/2}, y_{k/2+1}) \;=\; \{x : \hat{F}_n(x) \;=\; 1/2\}.$$

Of course, then $y_{n/2}$ and possibly $y_{n/2+1}$ will be included in the interval. Thus, in the general case we get

$$\hat{F}_n^-(1/2) \;=\; y_j \;=\; y_{n/2}, \quad \hat{F}_n^+(1/2) \;=\; y_{k/2+1}.$$

Solution to Exercise 1.2.23: Let $b_1 < b_2 < \ldots < b_m$ be the unique values of ϕ as in the canonical representation of ϕ (which is basically Exercise 1.2.21). Using the same notation as Exercise 1.2.21, $B_i = \phi^{-1}(\{b_i\})$. Let $A \subset I\!R$, and of course A should be a Borel set (but it doesn't matter). Since $\phi^{-1}(I\!R \setminus \{b_1, b_2, \ldots, b_m\}) = \emptyset$,

$$\mu\left(\phi^{-1}(A)\right) \;=\; \mu\left(\phi^{-1}(A \cap \{b_1, b_2, \ldots, b_m\})\right).$$

Since the latter set is finite, we may break it into its singletons and add their measures:

$$\begin{aligned} \mu\left(\phi^{-1}(A)\right) &= \sum_{i:b_i \in A} \mu\left(\phi^{-1}(\{b_i\})\right) \\ &= \sum_{i:b_i \in A} \mu(B_i). \end{aligned}$$

This gives a formula for computing $\mu\left(\phi^{-1}(A)\right)$, but we can give a more succinct formula for $\mu \circ \phi^{-1}$, namely

$$\mu \circ \phi^{-1} \;=\; \sum_{i=1}^{m} \mu(B_i)\delta_{b_i}.$$

Solution to Exercise 1.2.24: (a) $f(x) = x + a$ for some $a \in I\!R$. Clearly, f "shifts" or translates points x by the amount a. Since Lebesgue measure is an extension of "length" from intervals to fairly arbitrary sets (Borel sets), and "length" is unchanged by shifting, one would conjecture that $m \circ f^{-1}$ is also Lebesgue measure. To verify this, we use Theorem 1.1.3, which asserts that Lebesgue measure is unique with a certain property: let $-\infty < c < b < \infty$, then the induced measure of the closed interval $[c, b]$ is

$$\begin{aligned} (m \circ f^{-1})([c,b]) &= m(\{\, x \,:\, x + a \in [c,b] \,\}) \;=\; m(\{\, x \,:\, x \in [c-a, b-a] \,\}) \\ &= m([c-a, b-a]) \;=\; (b-a) - (c-a) \;=\; b - c\,. \end{aligned}$$

Thus, the induced $m \circ f^{-1}$ measure assigns to each finite closed interval its length, and so $m \circ f^{-1} = m$ by the uniqueness part of Theorem 1.1.3. This exercise shows that Lebesgue measure is *translation invariant*.

(b) Assume first $f(x) = ax$ for some $a > 0$. Clearly f magnifies or stretches things (or shrinks them, if $a < 1$). Multiplying a length by a is like changing units, and all "lengths" should be correspondingly changed. Let $[c, b]$ be a closed interval as in the solution to part (a). Note that $f^{-1}[c, b] = \{x : ax \in [c, b]\} = \{x : x \in [c/a, b/a]\}$. To see this latter set equality, note that $c \le ax \le b$ just in case $c/a \le x \le b/a$, where we are using positivity of a. Thus, we would conjecture that $m \circ f^{-1} = a^{-1}m$, i.e. $m \circ f^{-1}(B) = a^{-1}m(B)$. To show this, it suffices to show that $a(m \circ f^{-1}) = m$, for which we can use the uniqueness part of Theorem 1.1.3.

$$a(m \circ f^{-1})[c, b] \;=\; a \cdot m([c/a, b/a]) \;=\; a(b/a - c/a) \;=\; b - c\,.$$

Hence, $m \circ f^{-1} = a^{-1}m$. This exercise shows that Lebesgue measure is *scale equivariant*.

Now if $f(x) = ax$ for some $a < 0$, then the same argument applies except some things are "flipped around". In particular, $f^{-1}[c, b] = [b/a, c/a]$ because $a < 0$. However, following the same argument one sees that $|a|(m \circ f^{-1})[c, b] = b - c$, so $m \circ f^{-1} = |a|^{-1}m$.

In general then, we see that $m \circ f^{-1} = |a|^{-1}m$ when $f(x) = ax$ for $a \neq 0$.

(c) $f(x) = 0$ for all x. Note that $f^{-1}(B) = \{x : 0 \in B\} = \mathbb{R}$ if $0 \in B$ and otherwise $= \emptyset$. Thus,

$$(m \circ f^{-1})(B) \;=\; \begin{cases} m(\mathbb{R}) = \infty & \text{if } 0 \in B, \\ m(\emptyset) = 0 & \text{if } 0 \notin B. \end{cases}$$

Another way to write this is $m \circ f^{-1} = \infty \cdot \delta_0$.