Solutions to Homework 2

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Solution to Exercise 1.2.26: We want to show that

$$\int f d\# \; = \; \sum_i f(a_i) \quad .$$

Start with simple functions. Let

$$\phi \; = \; \sum_i b_i I_{B_i},$$

be in canonical form with all $b_i \geq 0$. Then

$$\int \phi d\# \; = \; \sum_i b_i \#(B_i).$$

Given any $a_j \in \Omega$, there is exactly one i such that $a_j \in B_i$, and then $\phi(a_j)$ $= b_i$. Thus,

$$b_i \#(B_i) \; = \; \sum_{j: a_j \in B_i} \phi(a_j).$$

Thus,

$$\int \phi d\# \; = \; \sum_i \sum_{j: a_j \in B_i} \phi(a_j) \; = \; \sum_j \phi(a_j).$$

Turning now to nonnegative functions $f \geq 0$, define simple functions

$$\phi_n(a) \; = \; \sum_{i=1}^n f(a_i) I_{\{a_i\}}.$$

Note that this is not generally the canonical form since the a_i's may not be distinct and $\bigcup_{i=1}^n \{a_i\}$ may not be all of Ω, but

$$\int \phi_n d\# \; = \; \sum_j \phi_n(a_j) \; = \; \sum_{i=1}^n f(a_i).$$

Note that if Ω is finite, say $\#(\Omega) = m$, then $\phi_n = f$ for all $n \geq m$, and we are done. So, assuming $\#(\Omega) = \infty$, then $0 \leq \phi_n \uparrow f$, so by MCT

$$\int f d\# \; = \; \lim_n \int \phi_n d\# \; = \; \lim_n \sum_{i=1}^n f(a_i) \; = \; \sum_{i=1}^\infty f(a_i).$$

Turning finally to the general case, $f = f_+ - f_-$, we have of course that $\int f_\pm d\# = \sum_i f_\pm(a_i)$. Assuming at least one of $\int f_\pm d\# < \infty$ so that $\int f d\#$ is defined, then

$$\int f_+ d\# - \int f_- d\# \ = \ \sum_i f_+(a_i) - \sum_i f_-(a_i) \ = \ \sum_i (f_+(a_i) - f_-(a_i)) \ = \ \sum_i f(a_i),$$

as claimed.

Solution to Exercise 1.2.27: We want to show that

$$\int f d\mu \ = \ \sum_i f(a_i)\mu(\{a_i\}) \quad ,$$

Start with nonnegative simple functions. Let

$$\phi \ = \ \sum_i b_i I_{B_i},$$

be in canonical form with all $b_i \geq 0$ and the B_i a partition of Ω. Then

$$\begin{aligned} \int \phi d\mu &= \sum_i b_i \mu(B_i) \\ &= \sum_i b_i \sum_{j:a_j \in B_i} \mu(\{a_j\}) \\ &= \sum_i \sum_{j:a_j \in B_i} b_i \mu(\{a_j\}) \\ &= \sum_i \sum_{j:a_j \in B_i} \phi(a_j)\mu(\{a_j\}) \\ &= \sum_{j:a_j \in \Omega} \phi(a_j)\mu(\{a_j\}) \quad . \end{aligned}$$

The first equation is the definition of the integral of a nonnegative simple function, the second follows since the B_i can be represented as the countable disjoint union of their elements, the third is trivial, the fourth since by the construction of the canonical representiation of a simple function, $a_j \in B_i \implies \phi(a_j) = b_i$, and the last equation since the B_i form a partition of Ω.

Turning now to nonnegative functions $f \geq 0$, let $0 \leq \phi_n \uparrow f$ be a sequence of simple functions. Then by MCT,

$$\begin{aligned} \int f d\mu &= \lim_{n\to\infty} \int \phi_n d\mu \\ &= \lim_{n\to\infty} \sum_i \phi_n(a_i)\mu(\{a_i\}). \end{aligned}$$

If we can interchange the limit and the summation, then we get

$$\begin{aligned} \int f d\mu &= \sum_i \lim_{n\to\infty} \phi_n(a_i)\mu(\{a_i\}) \\ &= \sum_i f(a_i)\mu(\{a_i\}) \quad , \end{aligned}$$

which is the desired result. Note by the previous exercise (which was shown in class) we can write

$$\sum_i \phi_n(a_i)\mu(\{a_i\}) \;=\; \int g_n d\#,$$

where $0 \le g_n(a_i) = \phi_n(a_i)\mu(\{a_i\}) \uparrow f(a_i)\mu(\{a_i\}) = g(a_i)$, and $\sum_i f(a_i)\mu(\{a_i\}) = \int g d\#$, so by MCT applied to counting measure on Ω, we have

$$\begin{aligned} \lim_{n\to\infty} \sum_i \phi_n(a_i)\mu(\{a_i\}) &= \lim_{n\to\infty} \int g_n d\# \\ &= \int g d\# \\ &= \sum_i f(a_i)\mu(\{a_i\}). \end{aligned}$$

This shows the desired result for $f \ge 0$.

Turning to general $f = f_+ - f_-$, of course the previous part applies to both of $f_\pm$:

$$\int f_\pm d\mu \;=\; \sum_i f(a_i)\mu(\{a_i\}).$$

Assume that $\int f d\mu$ is defined, i.e. that one of $\int f_\pm d\mu < \infty$, say $\int f_+ d\mu < \infty$. This implies that for all $a_i \in \Omega$, $f_+(a_i)\mu(\{a_i\}) < \infty$. If Ω is finite, say $\#(\Omega) = m$, then there is no $\infty - \infty$ possible in computing

$$\begin{aligned} \int f_+ d\mu - \int f_- d\mu &= \sum_{i=1}^m f_+(a_i)\mu(\{a_i\}) - \sum_{i=1}^m f_-(a_i)\mu(\{a_i\}) \\ &= \sum_{i=1}^m [f_+(a_i)\mu(\{a_i\}) - f_-(a_i)\mu(\{a_i\})] \\ &= \sum_{i=1}^m [f_+(a_i) - f_-(a_i)]\mu(\{a_i\}) \\ &= \sum_{i=1}^m f(a_i)\mu(\{a_i\}). \end{aligned}$$

In the first line, we know that the first summation is $< \infty$. In the second line, we know that the terms $f_+(a_i)\mu(\{a_i\})$ are all finite, and only $-\infty$ is possible, no $+\infty$ term. In the third line, it is possible that $f_+(a_i) = \infty$ (which would imply $\mu(\{a_i\}) = 0$, but then $f_-(a_i) = 0$.

If $\#(\Omega) = \infty$, then put $\lim_{m\to\infty}$ in front of all of the finite summations in the previous calculation. All of the finite sums are defined with no problem, and if any infinite quantities come from the limits, it will be only $-\infty$ since we are assuming $\int f_+ d\mu < \infty$.

Of course, the same argument applies (but with sign changes) if $\int f_- d\mu < \infty$.

Solution to Exercise 1.2.37: Let $g_m = \sum_{i=1}^m f_n$. Since the $f_n \geq 0$ a.e., it follows that the g_m are nonnegative and monotonically increasing a.e.. Hence, by the Monotone Convergence Theorem,

$$\begin{aligned} & \sum_{n=1}^{\infty} \int f_n \, d\mu \\ = & \lim_{m\to\infty} \sum_{n=1}^{m} \int f_n \, d\mu \\ = & \lim_{m\to\infty} \int \sum_{n=1}^{m} f_n \, d\mu \\ = & \lim_{m\to\infty} \int g_m \, d\mu \\ = & \int \lim_{m\to\infty} g_m \, d\mu \\ = & \int \sum_{n=1}^{\infty} f_n \, d\mu. \end{aligned}$$

Note that the first equality depends on linearity of the integral, and it is the third equality where the MCT is used.

Solution to Exercise 1.3.6: We will proceed without naming all the measurable spaces implied in Theorem 1.3.4. Let A and B be measurable sets in the ranges of g and h, respectively. Then,

$$\begin{aligned} P[g(X) \in A \,\&\, h(Y) \in B] &= P[X \in g^{-1}(A) \,\&\, Y \in h^{-1}(B)] \\ &= P[X \in g^{-1}(A)]P[Y \in h^{-1}(B)] \\ &= P[g(X) \in A]P[h(Y) \in B]. \end{aligned}$$

The first and third equalities follow from the definition of inverse images and the second from independence of X and Y. The result establishes independence of $g(X)$ and $h(Y)$.

Solution to Exercise 1.3.15: If we take ν to be counting measure on $\mathbb{Z}_+ = \{1, 2, \ldots\}$. Then it was shown in class that

$$\int g d\nu \;=\; \sum_{n=1}^{\infty} g(n).$$

Let $f : \mathbb{Z}_+ \times \mathbb{R} \to \bar{\mathbb{R}}$. Assuming $\int f d(\nu \times m)$ exists, then by Fubini's theorem,

$$\begin{aligned} \int f d(\nu \times m) &= \int_R \int_{\mathbf{Z}_+} f(n,x) d\nu(n) dx \quad (\star) \\ &= \int_{\mathbf{Z}_+} \int_R f(n,x) dx d\nu(n) \quad (\star\star). \end{aligned}$$

If we denote $f(n,x) = f_n(x)$, then the expressions in $(\star)$ and $(\star\star)$ and their equality may be written as

$$\int_R \sum_{n=1}^{\infty} f_n(x) dx \;=\; \sum_{n=1}^{\infty} \int_R f_n(x) dx.$$

Now, existence of the integral requires measurability $f : (\mathbb{Z}_+ \times \mathbb{R}, \mathcal{P}(\mathbb{Z}_+) \times \mathcal{B}) \longrightarrow \left(\bar{\mathbb{R}}, \bar{\mathcal{B}}\right)$, and that at least one of $\int f_\pm d(\nu \times m) < \infty$. Since $f_\pm \geq 0$, Fubini's theorem applies to each of them, and so a sufficient condition for interchange of summation and integration is at least one of

$$\int \sum f_{n,\pm}(x) dx \;<\; \infty.$$

In particular, if all $f_n \geq 0$ so that $f_{n,-} = 0$, we can interchange $\int$ and $\sum_{n=1}^{\infty}$. Thus, we have actually generalized what we got in Exercise 1.2.33 by using Fubini's theorem.

In Exercise 1.2.33 we assumed that each f_n was Borel measurable. We need to show that this implies f is measurable w.r.t. the product σ-field $\mathcal{P}(\mathbb{Z}_+) \times \mathcal{B}$. Assume $f(n, \cdot) = f_n$ is extended Borel for each n. If $B \in \bar{\mathcal{B}}$, then $f^{-1}(B) = \{(n,x) : f(n,x) \in B\} = \bigcup_n (\{n\} \times \mathbb{R}) \cap \{(n,x) : f(n,x) \in B\} = \bigcup_n \{n\} \times f_n^{-1}(B) \in \mathcal{P}(\mathbb{Z}_+) \times \mathcal{B}$. (Note: It is also easy to see that measurability of f implies measurability for each f_n, so the condition is necessary and sufficient.)

Solution to Exercise 1.3.18: It is easy to see

$$\begin{aligned} &\{(x_1, x_2, ..., x_n) \,:\, x_i = x_j \text{ for some } i \neq j\} \\ &= \bigcup_{i=1}^{n-1} \bigcup_{j=i+1}^{n} \{(x_1, x_2, ..., x_n) \,:\, x_i = x_j\}. \end{aligned}$$

If we set

$$B_{ij} \;=\; \{(x_1, x_2, ..., x_n) \,:\, x_i = x_j\},$$

then it suffices to show $m^n(B_{ij}) = 0$ for all $i < j$. For notational simplicity, we will take $i = 1$ and $j = 2$, then

$$\begin{aligned} m^n(B_{12}) &= \int I_{B_{12}} dm^n \\ &= \int_{R^{n-2}} \int_{R^2} I_{B_{12}}(x_1, x_2, \ldots, x_n) dm^2(x_1, x_2) dm^{n-2}(x_3, \ldots, x_n). \end{aligned}$$

In the last equation, we used the fact that we can represent $m^n = m^2 \times m^{n-2}$. Now for any fixed $(x_2, \ldots, x_n)$, the inner integral above may be written

$$\int_{R^2} I_{B_{12}}(x_1, x_2, \ldots, x_n) dm^2(x_1, x_2) \;=\; m^2(\{(x_1, x_2) : x_1 = x_2\} \;=\; 0,$$

where the last equality follows from Exercise 1.3.17. This shows $m^n(B_{12}) = 0$. If $i < j$ but $(i, j) \neq (1, 2)$, then we can appeal to an "obvious symmetry," or repeat the above argument with more complex notation (like writing $dm^{n-2}(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n)$ for the outer integral above).

Since Exercise 1.3.17 was not assigned, we provide its solution. Let $C = \{(x_1, x_2) : x_1 = x_2\}$. For any x_1, $I_C(x_1, x_2)$ is nonzero at only one value of x_2, namely, when $x_1 = x_2$. Thus, for any x_1, $m(\{x_2 : I_C(x_1, x_2) \neq 0\}) = 0$. Thus, by Fubini's theorem,

$$\begin{aligned} m^2(C) &= \int \int I_C(x_1, x_2) dx_2 dx_1 \\ &= \int m(\{x_2 : I_C(x_1, x_2) = 1\})\, dx_1 \\ &= \int 0\, dx_1 \\ &= 0. \end{aligned}$$