Solutions Homework 5

October 30, 2018

Solution to Exercise 2.4.4

(a) This is trivial by properties of the determinant: the determinant of a product is the product of the determinant, and taking transpose doesn't change the determinant. Thus, if $V = AA^t$, then

$$\det(V) \;=\; \det(AA^t) \;=\; \det(A)\det(A^t) \;=\; \det(A)^2.$$

Actually, since we could have $\det(A) < 0$, we can only conclude $\det(A) = |\det(V)^{1/2}|$. Of course, $\det(A) \neq 0$ if and only if $\det(V) \neq 0$.

(b) By Proposition 1.3.3 and Proposition 1.4.3, $\mathrm{Law}[\underline{Z}] \ll m^n$ and the Lebesgue density is

$$f_{\underline{Z}}(\underline{z}) \;=\; (2\pi)^{-n/2}\exp(-\underline{z}^t\underline{z}/2).$$

If we can find a matrix A such that $AA^t = V$, then we know from Exercise 2.3.3(b) that $A\underline{Z} + \underline{\mu} \sim N(\underline{\mu}, V)$. Assuming $\det(V) \neq 0$ and hence $\det(A) \neq 0$, the transformation $h : I\!\!R^n \longrightarrow I\!\!R^n$ is a bijection with inverse $h^{-1}(\underline{y}) = A^{-1}(\underline{y} - \underline{\mu})$ and derivative $Dh^{-1}(\underline{y}) = A^{-1}$ so the Jacobian is $|\det(A^{-1})| = \det(V)^{-1/2}$. Thus, by Proposition 2.4.2, if $\underline{Y} = A\underline{Z} + \underline{\mu} \sim N(\mu, V)$ then the Lebesgue density for $\underline{Y}$ is given by

$$\begin{aligned}
f_{\underline{Y}}(\underline{y}) &= \det(V)^{-1/2} f_{\underline{Z}}\left(A^{-1}(\underline{y} - \underline{\mu})\right) \\
&= (2\pi)^{-n/2}\det(V)^{-1/2}\exp\left[\left(A^{-1}(\underline{y} - \underline{\mu})\right)^t \left(A^{-1}(\underline{y} - \underline{\mu})\right)\right] \\
&= (2\pi)^{-n/2}\det(V)^{-1/2}\exp\left[(\underline{y} - \underline{\mu})^t \left(A^{-1}\right)^t A^{-1}(\underline{y} - \underline{\mu})\right] \\
&= (2\pi)^{-n/2}\det(V)^{-1/2}\exp\left[(\underline{y} - \underline{\mu})^t \left(AA^t\right)^{-1} (\underline{y} - \underline{\mu})\right] \\
&= (2\pi)^{-n/2}\det(V)^{-1/2}\exp\left[(\underline{y} - \underline{\mu})^t V^{-1}(\underline{y} - \underline{\mu})\right].
\end{aligned}$$

This gives the desired Lebesgue density for the $N(\underline{\mu}, V)$ distribution.

There does remain one detail: how do we know there exists a matrix A such that $AA^t = V$? Well, there are many such matrices, but one obvious one is $V^{1/2}$ which is defined as follows. Let $V = U\Lambda U^t$ be the spectral decomposition of V (Theorem 2.1.6). Here, $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ where the diagonal entries of Λ are the eigenvalues of V, which are nonnegative since V is nonnegative definite. Defining $\Lambda^{1/2}$ by $\mathrm{diag}(\lambda_1^{1/2}, \ldots, \lambda_n^{1/2})$, then it is easy to check that $V^{1/2} = U\Lambda^{1/2}U^t$ is a symmetric matrix and $(V^{1/2})^2 = V$. Thus, we may take $A = V^{1/2}$.

Solution to Exercise 2.4.6: Note that T ranges over positive and negative real numbers, so we may restrict attention to m being an integer (else we would have to deal with complex numbers). Clearly $E[T^m]$ for m even, but it may be ∞. Now we just try to do the integral:

$$E[T^m] \quad = \quad \int_{-\infty}^{\infty} t^m \frac{\Gamma((\nu+1)/2)}{\sqrt{\nu\pi}\,\Gamma(\nu/2)} (1 + t^2/\nu)^{-(\nu+1)/2}\, dt$$

In order for the integral to be finite, we need

$$\int_0^{\infty} \frac{t^m}{(1+t^2/\nu)^{(\nu+1)/2}}\, dt \ < \ \infty.$$

If $m < 0$ the integrand blows up at the origin. If $m \geq 0$, then the problematic part of the integral is the large values of t. Note that as $t \to \infty$, $t^m/[(1+t^2/\nu)^{(\nu+1)/2}]$ behaves like $t^{m-\nu-1}$. Put somewhat more formally,

$$\exists C_1, C_2 \in (0,\infty)\, \forall t > 1, \quad C_1 t^{m-\nu-1} \ \leq \ \frac{t^m}{(1+t^2/\nu)^{(\nu+1)/2}} \ \leq \ C_2 t^{m-\nu-1}.$$

Now,

$$\int_1^{\infty} t^{m-\nu-1}\, dt \ < \ \infty \text{ if and only if } m - \nu - 1 < -1 \text{ if and only if } m < \nu.$$

Thus, for $m \geq 0$, we conclude that that $E[T^m]$ is finite (and hence exists) if and only if $m < \nu$, and if $m \geq \nu$ and m is even, then the integral exists but is infinite. If $m \geq \nu$ and m is odd, then t^m has the same sign as t, and we will get an $\infty - \infty$ when we try to do the integral, so it will be undefined.

Recall that if $Z \sim N(0,1)$ and $V \sim \chi^2_\nu$ are independent, then

$$T \ = \ Z/(V/\nu)^{1/2},$$

has a t-distribution with ν d.f. Clearly, then we have

$$\begin{aligned} E[T^m] \quad &= \quad \nu^{m/2} E[Z^m/V^{m/2}] \\ &= \quad \nu^{m/2} E[Z^m] E[V^{-m/2}], \end{aligned}$$

provided everything is finite, i.e. $0 \le m < \nu$. Note that the sign of the exponent of ν given in the text is incorrect. Now, we just have to check if the formula holds if is even but $m < 0$ or $m \ge \nu$, i.e., make sure the r.h.s. if ∞ in these cases. If $m < 0$ and even , then $E[Z^m] = \infty$ and $E[V^{-m/2}] > 0$, so the r.h.s. with be ∞. If $m \ge \nu$ and m is even, then $E[Z^m] > 0$. The important factors in $v^{-m/2} f_V(v)$ will for v near 0 the powers of v, (the exponential $e^{-v/2}$ is bounded away from 0 and ∞ for v near 0), i.e. $v^{-m/2+\nu/2-1}$, and this will lead to an infinite integral (on say, $(0,1)$) if the exponent is ≤ -1, i.e. $m \ge \nu$. This completes the exercise.

Solution to Exercise 3.1.12 Let

$$\bar{F}(z) \;=\; \int_z^\infty \frac{x^{\alpha-1}}{\Gamma(\alpha)}\, e^{-x}\, dx.$$

There are a lot of integration by parts games we could play here, but in general we should separate the power function $x^{\alpha-1}$ from the exponential e^{-x} because we can't get a simple, explicit form for an indefinite integral, otherwise. So let's start with

$$\begin{aligned} u &= x^{\alpha-1} & dv &= e^{-x}\, dx, \\ du &= (\alpha-1)x^{\alpha-2}\, dx & v &= -e^{-x}. \end{aligned}$$

We obtain

$$\begin{aligned} \bar{F}(z) &= [\Gamma(\alpha)]^{-1}\, x^{\alpha-1}(-e^{-x})\Big|_{x=z}^{\infty} - [\Gamma(\alpha)]^{-1} \int_z^\infty (\alpha-1)x^{\alpha-2}(-e^{-x})\, dx \\ &= [\Gamma(\alpha)]^{-1}\, z^{\alpha-1}e^{-z} + [\Gamma(\alpha)]^{-1}\, (\alpha-1) \int_z^\infty x^{\alpha-2}e^{-x}\, dx. \end{aligned}$$

For $x \ge z$, $x^{\alpha-2} \le x^{\alpha-1}/z$, so

$$\begin{aligned} \int_z^\infty x^{\alpha-2}e^{-x}\, dx &\le \frac{1}{z} \int_z^\infty x^{\alpha-1}e^{-x}\, dx \\ &= \frac{1}{z}\Gamma(\alpha)\bar{F}(z). \end{aligned}$$

Plugging this into our previous calculation and remebering that $\Gamma(\alpha)/\Gamma(\alpha-1) = \alpha - 1$ is just an unimportant constant, we get

$$\bar{F}(z) \;=\; [\Gamma(\alpha)]^{-1}\, z^{\alpha-1}e^{-z} \;+\; \mathcal{O}(1/z)\bar{F}(z).$$

The $\mathcal{O}(1/z)$ is of course as $z \to \infty$. Now we subtract the last term from both sides to obtain

$$\bar{F}(z)\,[1 + \mathcal{O}(1/z)] \;=\; [\Gamma(\alpha)]^{-1}\, z^{\alpha-1} e^{-z}.$$

Now we apply the same argument as in the normal tail example (see the discussion of (3.8) from (3.7)) to conclude

$$\bar{F}(z) \;=\; [\Gamma(\alpha)]^{-1}\, z^{\alpha-1} e^{-z}\,[1 + \mathcal{O}(1/z)]\,.$$

Interestingly, the tail area in the Gamma distribution is asymptotically equivalent to the density.

Solution to Exercise 3.2.8: We can solve this fairly easily using Proposition 3.2.8 by computing second moments. Since the X_i's are assumed to have mean 0 and finite variance $\sigma^2 > 0$, we have (using the assumed independence of the X_i's in the variance calculation)

$$\begin{aligned} E[Y_n] &= 0, \\ \mathrm{Var}[Y_n] &= \sigma^2 \sum_{i=1}^{n} i^{2p}. \end{aligned}$$

Now we employ the simple trick from analysis of bounding the sum by a pair of integrals (with possibly additional terms). There are a few cases to consider.

$p \geq 0$**:** In this case, the mapping $x \mapsto x^{2p}$ is nondecreasing, so

$$x^{2p} I_{[i-1,i]}(x) \;\leq\; i^{2p} I_{[i-1,i]}(x) \;\leq\; x^{2p} I_{[i,i+1]}(x).$$

Adding up over $1 \leq i \leq n$ and integrating gives

$$\frac{n^{2p+1}}{2p+1} \;\leq\; \int_0^n x^{2p}dx \;\leq\; \sum_{i=1}^{n} i^{2p} \;\leq\; \int_1^{n+1} x^{2p}dx \;\leq\; \frac{(n+1)^{2p+1}-1}{2p+1}.$$

Since $n^{2p+1} \to \infty$, it is easy to see that

$$\frac{(n+1)^{2p+1}-1}{2p+1} \;=\; O(n^{2p+1}).$$

We do not really need the lower bound, but using it we can check that in fact $E[Y_n^2] \sim \sigma^2 n^{2p+1}/(2p+1)$. To see the claim in the last equation, note that the term $-1/(2p+1) = o(n^{2p+1})$ and since $[(n+1)/n]^{2p+1} \to 1$, we also have $(n+1)^{2p+1} = O(n^{2p+1})$. So, we have

$$E[Y_n^2] \;=\; O(n^{2p+1})$$

and this implies by the proposition that

$$Y_n \;=\; O_P(n^{p+1/2}).$$

$-1/2 < p < 0$: In this case, the map is decreasing, so an appropriate upper bound is

$$i^{2p} I_{[i,i+1]}(x) \;\le\; x^{2p} I_{[i,i+1]}(x).$$

Adding up and integrating again, we get

$$\sum_{i=1}^{n} i^{2p} \;\le\; \int_1^n x^{2p} dx \;=\; \frac{n^{2p+1}-1}{2p+1} \;=\; O(n^{2p+1}).$$

Then from the proposition we get $Y_n = O_P(n^{p+1/2})$.

$p = -1/2$: For this case, the same upper bound from the previous case applies, but the integral is now

$$\int_1^n x^{-1} dx \;=\; \log(n)$$

and so we get $Y_n = O_P((\log n)^{1/2})$.

$p < -1/2$: The same upper bound as the previous two cases applies, but now the integral is bounded as $n \to \infty$:

$$\int_1^n x^{2p} dx \;=\; \frac{1 - x^{2p+1}}{-(2p+1)} \;\to\; 1/(-(2p+1)),$$

since of course $2p+1 < 0$. Hence, $E[Y_n^2]$ is bounded and so is $O(1)$, so $Y_n = O_P(1)$.