Stat650: Homework 4

April 7, 2010

Exercise 1 Here, we develop a general theory of integrating factors for classical Ito equations. We consider a one-dimensional equation of the form

$$dX(t) \;=\; b(t, X(t))dt \,+\, \sigma(t, X(t))dW(t), \tag{1}$$

where $W(t)$ is a standard Brownian motion. The goal is to find a factor $f(t, x)$ such that we can multiply both sides of the equation by $f(t, X(t))$, add some terms to both sides, recognize the left hand side of the result as the differential of a product, and eliminate the dependence of the right hand side on $X(t)$. Then we can simply integrate both sides and obtain a "solution" in the form

$$X(t) \;=\; \frac{1}{f(t, W(t))}\left\{f(0)X(0) \,+\, \int_0^t u(s, W(s))ds \,+\, \int_0^t v(s, W(s))dW(s)\right\},$$

for some functions u and v.

We will use the first year calculus notation when we write partial derivatives. For example, $f(t, w)$ is a function of 2 variables. $\frac{\partial f}{\partial w}$ denotes partial differentiation w.r.t. the second variable.

Hopefully, I have done all the calculus/algebra correctly.

(a) Given that $X(t)$ is a solution of the SDE (1) and $f(t, w)$ is a suitably smooth function, show that

$$\begin{aligned} & d\left[f(t, W(t))X(t)\right] \\ =\;& f(t, W(t))dX(t) \,+\, \left\{\frac{\partial f}{\partial t}(t, W(t))X(t) \,+\, \frac{1}{2}\frac{\partial^2 f}{\partial w^2}(t, W(t))X(t)\right. \\ & \left. \qquad + \frac{\partial f}{\partial w}(t, W(t))\sigma(t, X(t))\right\} dt \,+\, \frac{\partial f}{\partial w}(t, W(t))X(t)dW(t). \end{aligned}$$

Hence,

$$\begin{aligned} & d\left[f(t, W(t))X(t)\right] \\ =\;& \left\{f(t, W(t))b(t, X(t)) \,+\, \frac{\partial f}{\partial t}(t, W(t))X(t)\right. \end{aligned}$$

$$
\begin{aligned}
&+ \frac{1}{2}\frac{\partial^2 f}{\partial w^2}(t, W(t))X(t) \; + \frac{\partial f}{\partial w}(t, W(t))\sigma(t, X(t))\Big\} \, dt \\
&+ \Big\{ f(t, W(t))\sigma(t, X(t)) \; + \; \frac{\partial f}{\partial w}(t, W(t))X(t) \Big\} \, dW(t).
\end{aligned}
$$

Now we want to choose $f(t, W(t))$ so as to make the last expression functionally independent of $X(t)$. This will also force $b(t, x)$ and $\sigma(t, x)$ to be of specific forms.

(b) From the factor in the last expression in part (a) in front of the $dW(t)$, show that in order to make it functionally independent of $X(t)$ (but it can still depend on $(t, W(t))$), $\sigma(t, x)$ must take the form

$$\sigma(t, x) \;=\; \alpha(t) + \beta(t)x,$$

for some functions $\alpha(t)$ and $\beta(t)$, and then our integrating factor must take the form

$$f(t, w) \;=\; \exp\left[\gamma(t) - \beta(t)w\right],$$

where $\gamma(t)$ is a fairly arbitrary function.

(c) From the factor in the last expression in part (a) in front of the $dW(t)$, show that in order to make it functionally independent of $X(t)$, the $\beta(t)$ must be a constant, say β, and $b(t, x)$ must take the form

$$b(t, x) \;=\; c(t) \;+\; r(t)x,$$

for some functions $c(t)$ and $r(t)$. Then, the $\gamma(t)$ from part (b) is given by (up to an arbitrary constant which is irrelevant)

$$\gamma(t) \;=\; \frac{1}{2}\beta^2 t \;-\; \int_0^t r(s)ds.$$

(d) Apply the above to give a general "solution" (in the sense described above) to equations of the form

$$dX(t) \;=\; [c(t) + rX(t)]dt \;+\; [\alpha(t) + \beta X(t)]dW(t),$$

where $c(t)$ and $\alpha(t)$ are given functions of time, and r and β are given constants.

Exercise 2 Continuing in the same vein as Exercise 1, we consider an equation of the form

$$dX(t) \;=\; b(t, X(t))dt \;+\; \beta(t)X(t)dW(t). \tag{2}$$

Now define an "integrating factor" of the form

$$F(t) \;=\; \exp\left[-\int_0^t \beta(s)dW(s) \;+\; \frac{1}{2}\int_0^t \beta^2(s)ds\right].$$

(a) From (2) show that

$$d(F(t)X(t)) \;=\; F(t)b(t, X(t))dt.$$

Thus, if we define

$$Y(t) \;=\; F(t)X(t),$$

so that $X(t) = Y(t)/F(t)$, then

$$dY(t) \;=\; F(t)f(t, Y(t)/F(t)).$$

This is an ordinary differential equation for $Y(t)$. In principle, we can compute a sample path of $W(t)$, compute the corresponding sample path of $F(t)$, plug it into this last equation, and "solve."

(b) Apply the above to solve

$$dX(t) \;=\; \frac{1}{X(t)}dt \;+\; \beta X(t)dW(t),$$

where β is a constant, and $X(0) > 0$.

Exercise 3 Ito's lemma also allows us to use "substitution" methods to solve SDEs. Consider the SDE

$$dX(t) \;=\; rX(t)dt \;+\; \beta X(t)dW(t).$$

Make the substitution $Y(t) = \log X(t)$ and obtain a solution. Compare with the result you get applying the formula you got from Exercise 1(d).

Exercise 4 **(a)** Let W be m-dimensional standard Brownian motion, where $m > 2$. Fix $R > 0$ and define the annulus

$$A_k \;=\; \{x \in I\!R^m : R < |x| < 2^k R\}.$$

Let τ_k be the first exit time of $W(t)$ from A_k:

$$\tau_k \;=\; \inf\{t \geq 0 : W(t) \notin A_k\}.$$

Show that for $x \in A_k$,

$$P[|W(\tau_k)| \;=\; R|W(0) = x]R^{2-m} + P[|W(\tau_k)| \;=\; 2^k R|W(0) = x](2^k R)^{2-m} \;=\; |x|^{2-m}.$$

Use this to derive a formula for $P[|W(\tau_k)| \;=\; R|W(0) = x]$.

(b) Show that as $k \to \infty$, $P[|W(\tau_k)| \;=\; R|W(0) = x] \to 0$. Use this to show that Brownian motion is transient in 3 or higher dimensions.